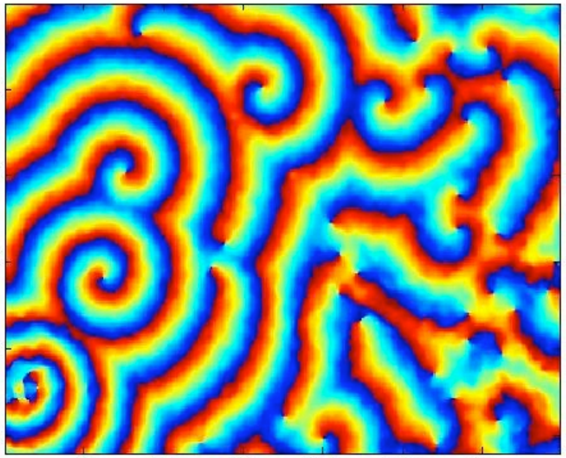
Wave pattern selection in excitable media

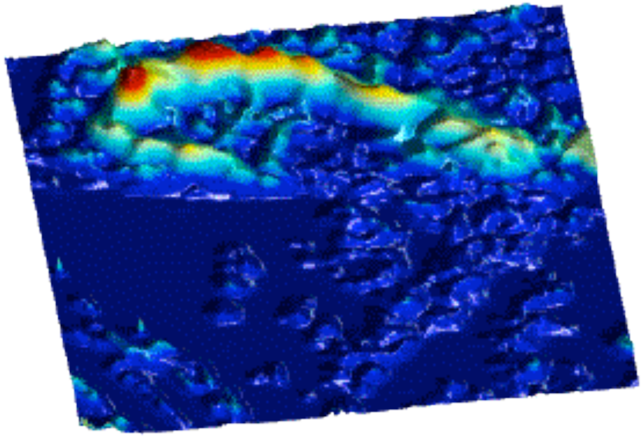
V.S. Zykov and E. Bodenschatz, MPI Goettingen

- Introduction
- Stabilized segments of TT waves
- Spiral selection for TT waves
- Stabilized segments of TP waves
- Spiral selection for TP waves
- Summary

Rotating spiral waves and wave segments

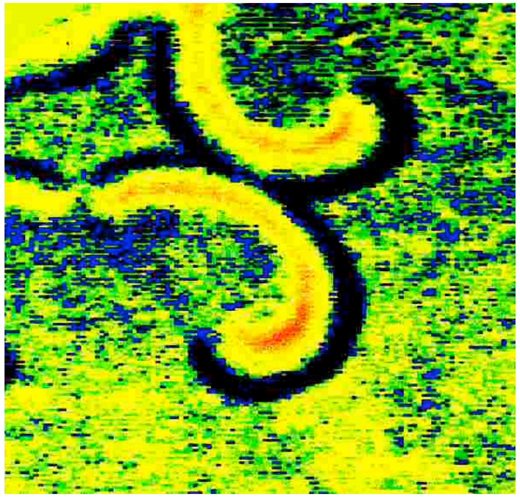


Aggregation of Dictyoselium discoideum



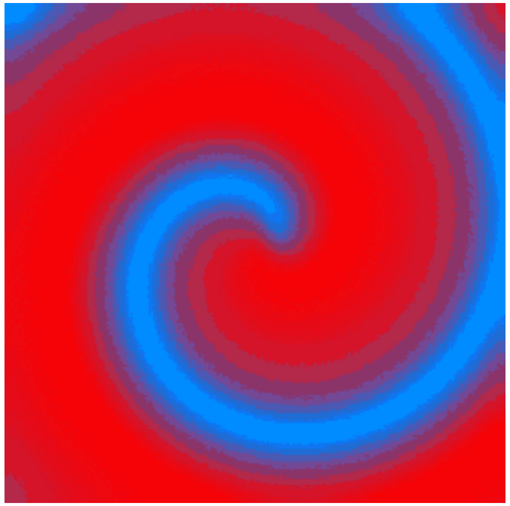
Electrical activity in cardiac tissue

The Belousov-Zhabotinsky reaction



NADH waves during the glycolysis

Simulation of wave processes

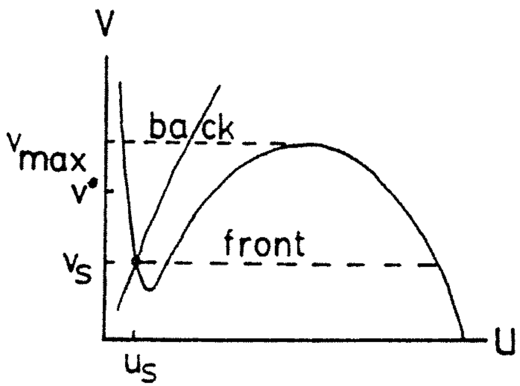


D. Barkley (1991)

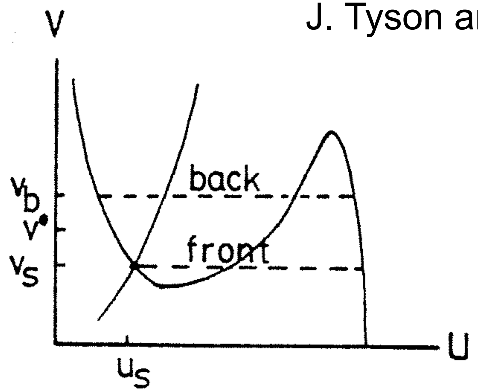
$$\frac{\partial u}{\partial t} = F(u,v) + D\nabla^2 u,$$

$$\frac{\partial v}{\partial t} = \varepsilon\, G(u,v)$$

J. Tyson and J. Keener (1988)



TP wave

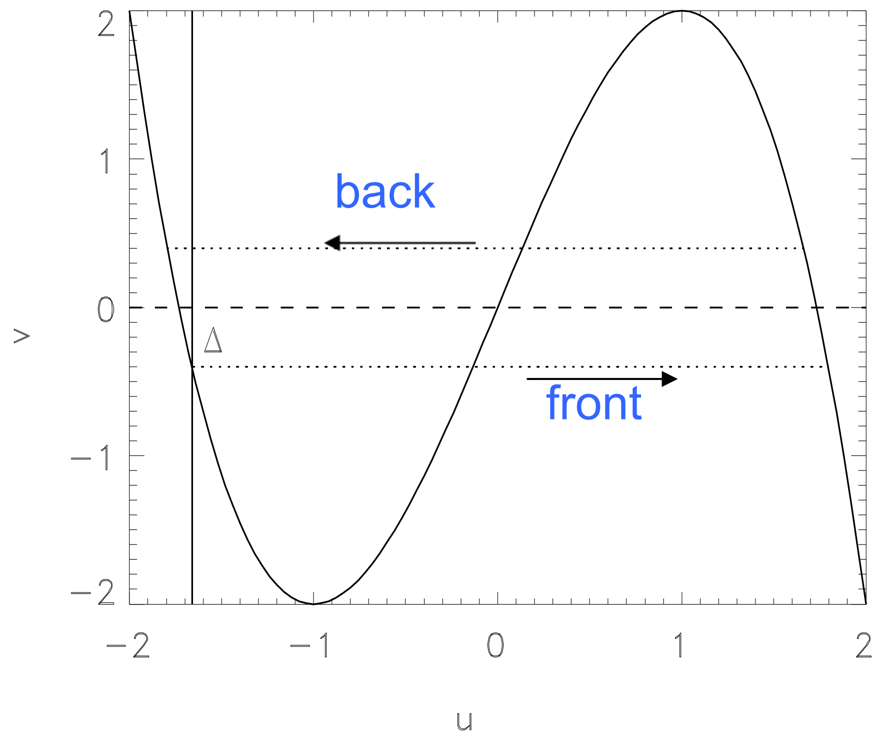


TT wave

The FitzHugh-Nagumo model

$$\frac{\partial u}{\partial t} = D\nabla^2 u + F(u,v),$$

$$\frac{\partial v}{\partial t} = \varepsilon G(u,v),$$



$$F(u,v) = 3u - u^3 - v,$$

$$G(u,v) = u - \delta$$

$$\delta = -1.68, \quad \varepsilon << 1$$

$$F(u,v) \approx -v - 6(u+\sqrt{3}), \quad |u+\sqrt{3}| << 1$$

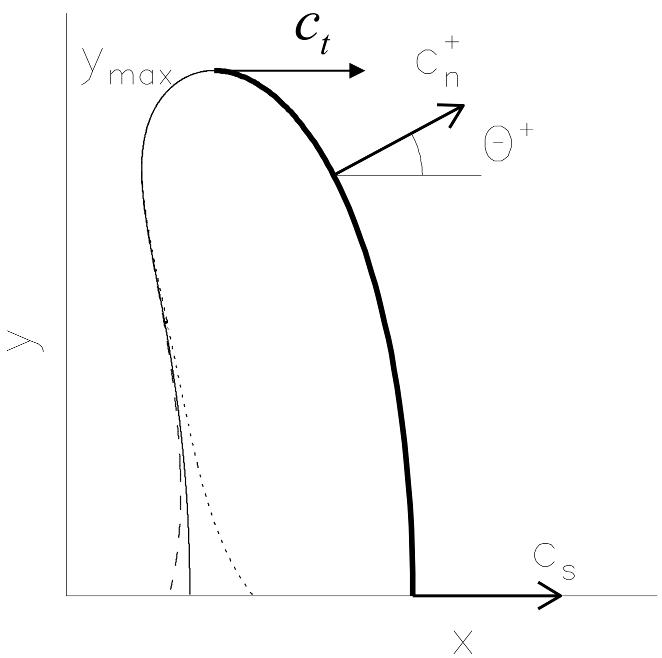
$$F(u,v) \approx -v - 6(u-\sqrt{3}), \quad |u-\sqrt{3}| << 1$$

$$v^* = 0, \quad \Delta = v^* - v_0, \ G^* = G(u(v^*), v^*)$$

$$c_p(v) = \alpha\sqrt{D}\Delta \equiv \sqrt{\frac{D}{2}}\Delta$$

$$d_u = \frac{2\Delta}{G^* \varepsilon}$$

Kinematical model of a stabilized wave segment



$$c_t = c_s = c_0 - Dk_m$$

$$c_n = c_p(v) - Dk$$

$$k^{\pm} = -d\Theta^{\pm} / ds$$

$$c_0 - Dk^+ = c_t \cos(\Theta^+)$$

$$c_p(v^-) - Dk^- = c_t \cos(\Theta^-)$$

$$c_t dv / dx = -\varepsilon G^*$$

"Critical finger" Karma, PRL, 1991; " Wave segment" Zykov, Showalter, PRL, 2005

Front of the wave segment

$$D\frac{d\Theta^+}{ds} = c_t\cos(\Theta^+) - c_0$$

$$dy^+ = -ds\cos(\Theta^+), \quad dx^+ = ds\sin(\Theta^+)$$

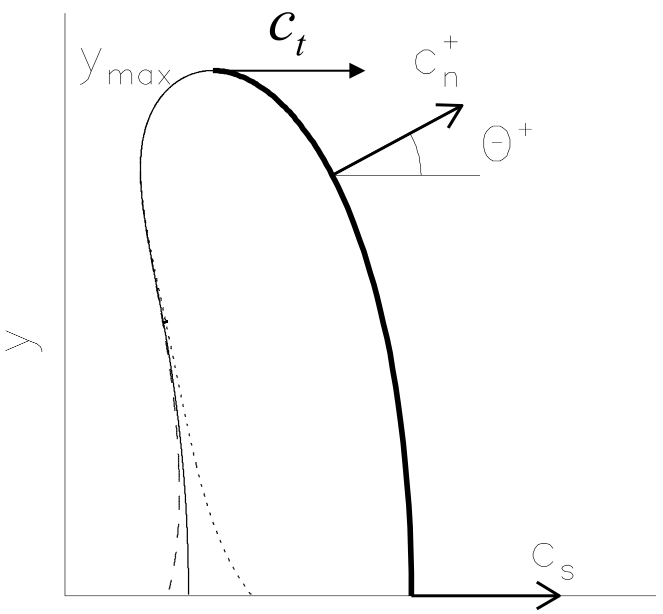
$$\frac{dx^+}{d\Theta^+} = \frac{D\sin(\Theta^+)}{c_t\cos(\Theta^+) - c_0}, \qquad \frac{dy^+}{d\Theta^+} = -\frac{D\cos(\Theta^+)}{c_t\cos(\Theta^+) - c_0}$$

$$\frac{x^+}{D} = \frac{1}{c_t}\ln\frac{c_0}{c_0 - c_t\cos(\Theta^+)},$$

$$\frac{y^+}{D} = -\frac{\Theta^+}{c_t} + \frac{2c_0}{c_t\sqrt{c_0^2 - c_t^2}}\arctan\frac{(c_0 + c_t)\tan(\Theta^+/2)}{\sqrt{c_0^2 - c_t^2}}.$$

Back of the wave segment

$C_t \equiv c_t / c_0 = 0.9598$



$B = 0.5110$ $B = 0.5101$

$B = 0.5108$

$$v^- = v^+ + \frac{G^* \varepsilon}{c_t}[x^+(y^+) - x^-(y^+)]$$

$$D\frac{d\Theta^-}{ds} = -c_0 + \frac{G^* \varepsilon \alpha}{c_t}(x^+ - x^-) + \cos(\Theta^-)$$

$$S = c_0 s / D, \quad X^\pm = c_0 x^\pm / D, \quad K = Dk / c_0$$

$$\frac{d\Theta^-}{dS} = \frac{B(X^+ - X^-)}{C_t} - 1 + C_t \cos(\Theta^-)$$

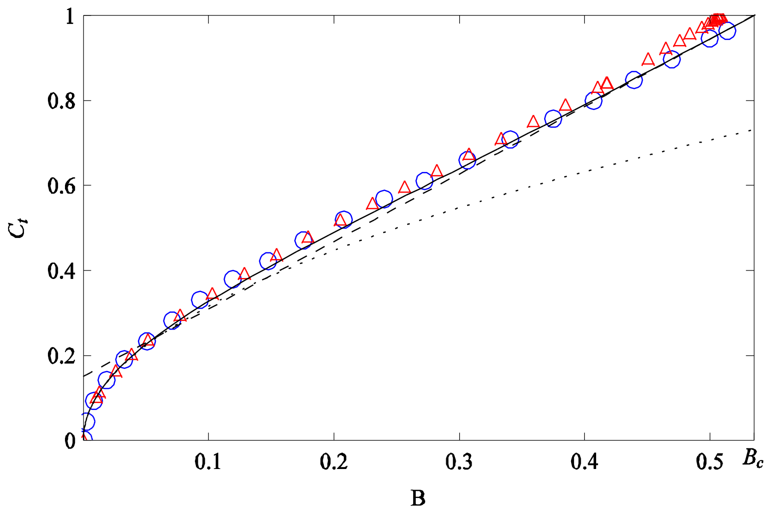
$$B = \frac{G^* \varepsilon}{\alpha^2 \Delta^3} \quad \text{alternatively} \quad B = \frac{2D}{d_u c_0^2}$$

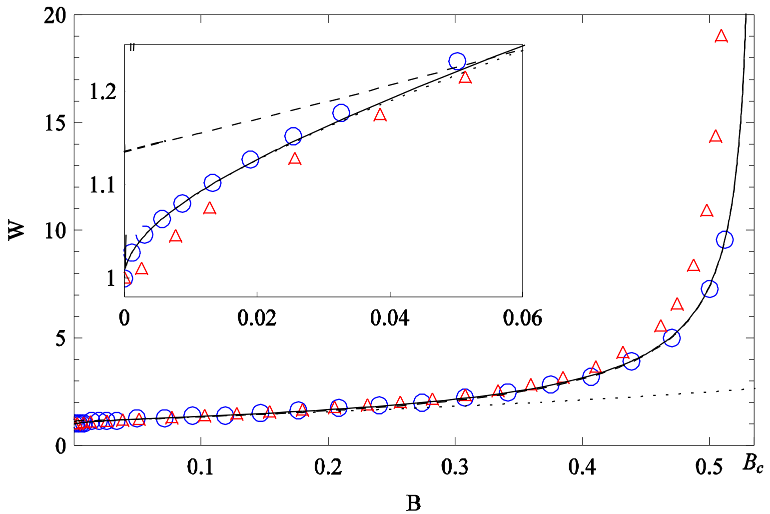
$$\Theta^- = \pi / 2, X^- = 0; \quad Y^- = Wc_0 / D, S = 0$$

$$dX^- / dS = \sin(\Theta^-), \quad dY^- / dS = -\cos(\Theta^-)$$

$$Y^- = 0, \Theta^- = \pi$$

Selected values





$$C_t = 1-(B_c - B)/0.63, \quad 0.1 < B < B_c = 0.535$$

$$C_t = \sqrt{B}, \qquad 0 < B < 0.1$$

$$C_t = \sqrt{B + (1-B_c)\left(\frac{B}{B_c}\right)^{\frac{5}{2}}}, \qquad 0 < B < B_c$$

$$W = \frac{2}{C_t\sqrt{1-C_t^2}} \arctan\frac{(1+C_t)}{\sqrt{1-C_t^2}} - \frac{\pi}{2C_t}.$$

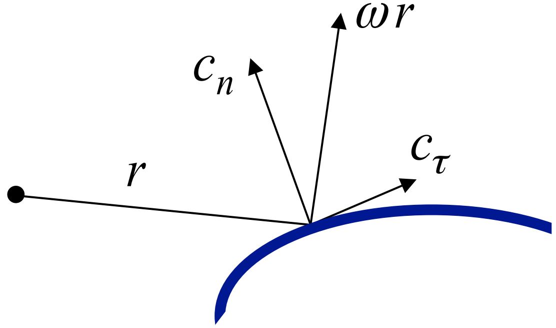
$$W = 1 + \frac{\pi}{4}\sqrt{B} + O(B), \quad B << B_c.$$

Kothe, Zykov, Engel, PRL, 2009

Kinematics of a rigidly rotating curve

$$\frac{dc_n}{ds} = \omega + kc_\tau$$

$$\frac{dc_\tau}{ds} = -kc_n$$



$$c_n = c_p - Dk$$

$$k = k(s), \qquad \alpha(s) = \alpha_0 - \int_0^s k ds'$$

$$x(s) = x_0 + \int_0^s \cos\alpha \, ds', \quad y(s) = y_0 + \int_0^s \sin\alpha \, ds'$$

Zykov, 1980

Kinematics of a wave front

Scaling

$$C = c / c_p, \quad S = s c_p / D$$

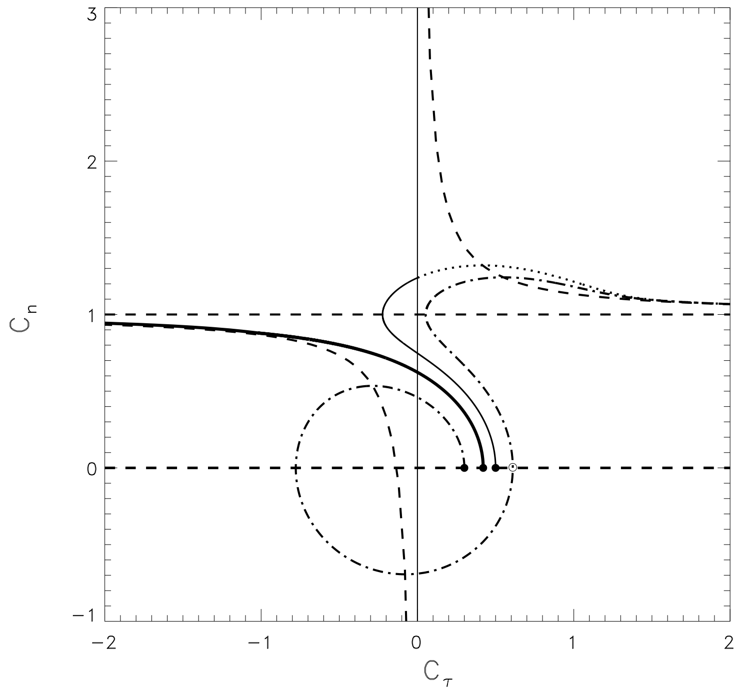
$$K = kD / c_p, \quad \Omega = \omega D / c_p^2$$

Dimensionless equations

$$\frac{dC_n}{dS} = \Omega + K C_\tau$$

$$\frac{dC_\tau}{dS} = -K C_n$$

$$C_n^+ = 1 - K^+$$

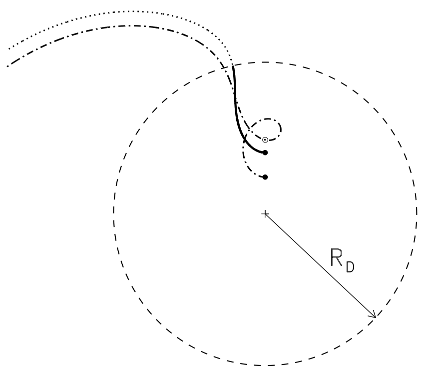


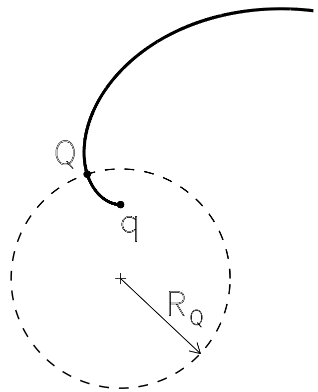
$$\Omega = 0.1333$$

$$C_n^+(0) = 0$$

$$C_\tau(0) = C_t$$

$$C_t = 0.4206$$





Zykov, 2008

Kinematics of a wave back

$$\frac{dC_n^-}{dS} = \Omega + KC_\tau$$

$$\frac{dC_\tau}{dS} = -KC_n^-$$

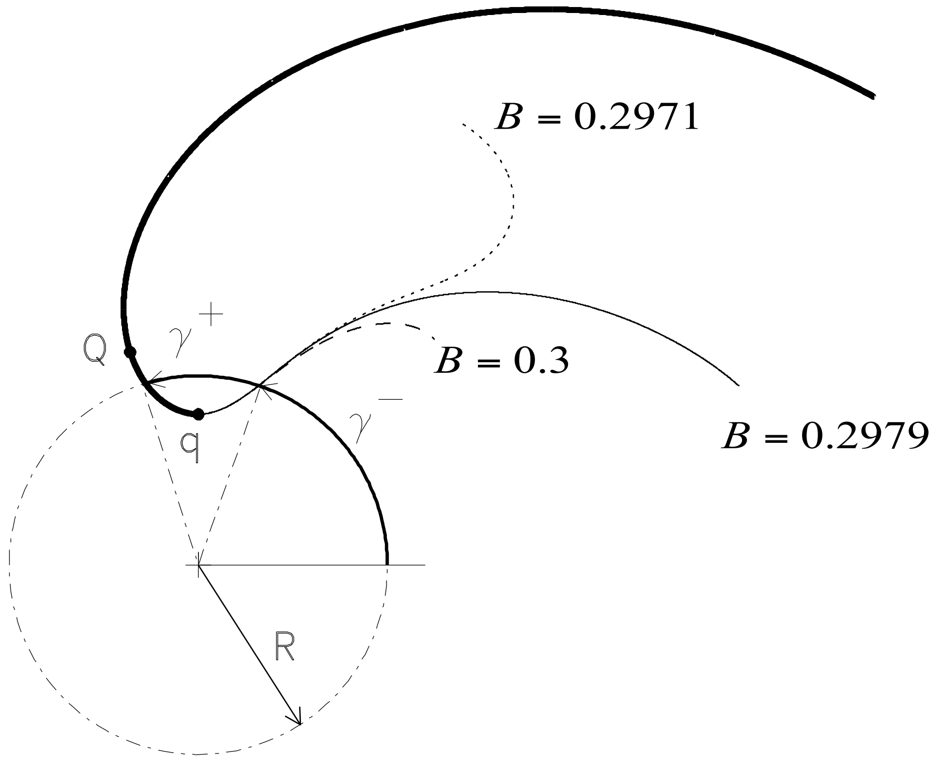
$$C_n^- = 1 - K^- - \frac{B}{\Omega}\left[\gamma^+(R) - \gamma^-(R)\right]$$

$$\Omega = 0.1333$$

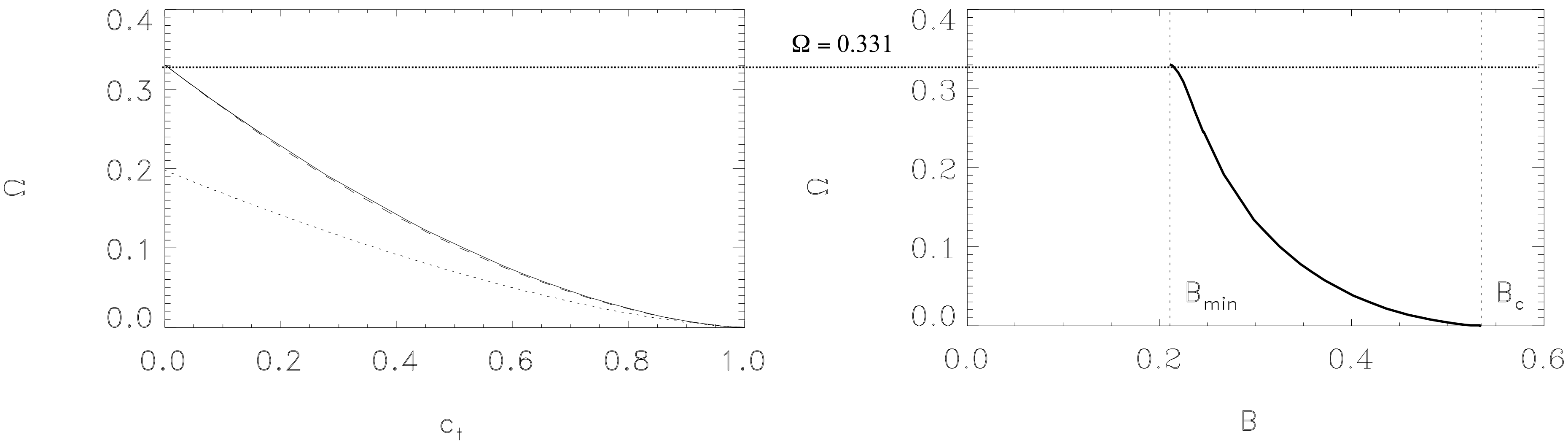
$$C_n^-(0) = 0$$

$$C_\tau^-(0) = C_t$$

$$v^- = v^+ + \frac{1-v^*}{\omega}\left[\gamma^+(r) - \gamma^-(r)\right]$$



Two selected relationships



$$\Omega = 0.198(1-C_t)^{3/2} + 0.133(1-C_t)^2$$ Zykov, 2008

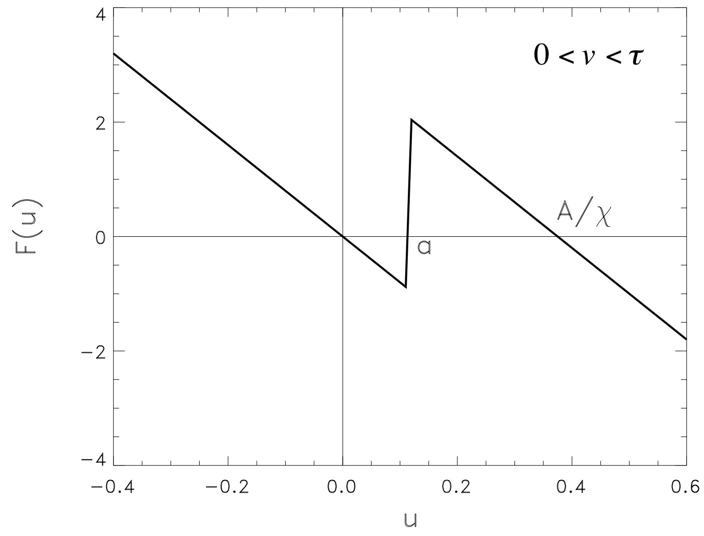
$$\Omega = \Omega_{FB}(B)$$

$$\Omega = 0.198(1-C_t)^{3/2}$$ Hakim and Karma, 1999

The Kessler-Levine model

$$\frac{\partial u}{\partial t} = F(u,v) + D\nabla^2 u,$$

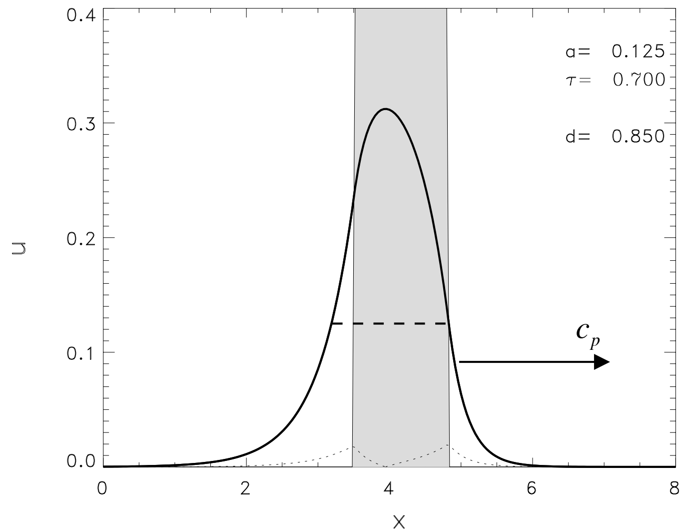
$$\frac{\partial v}{\partial t} = G(u,v),$$



$$F(u,v) = -\chi u + A\,H(u-a)[1 - H(v-\tau)],$$

$$G(u,v) = H(v)$$

$$v_{\min} << \tau < \tau_{aref} < \tau_{rref}$$



$$a = \left[a_{\max} - \eta \frac{v - \tau_{aref}}{v} \right],$$

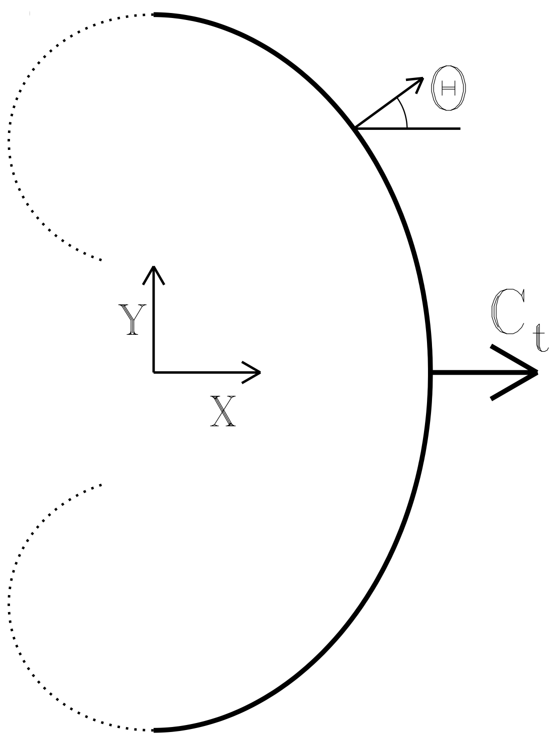
$$\eta = (a_{\max} - a_{\min}) \frac{\tau_{rref}}{\tau_{rref} - \tau_{aref}}$$

$$c_p = \sqrt{\chi D}(A/\chi - 2a) / \sqrt{a(A/\chi - a)}$$

D. Kessler and H. Levine (1993); H. Levine, I. Aranson, L. Tsimring, and T. Truong (1996)

Kinematical model of a stabilized wave segment in the KL model

$$C_t = c_t / c_0, \quad X = xc_0 / D, \quad K = kD / c_0$$



$$C_t = 1 - K_m$$

$$C_n = 1 - K$$

$$\frac{d\Theta}{dS} = C_t \cos(\Theta) - 1$$

$$\frac{dX}{d\Theta} = \frac{\sin(\Theta)}{C_t \cos(\Theta) - 1}$$

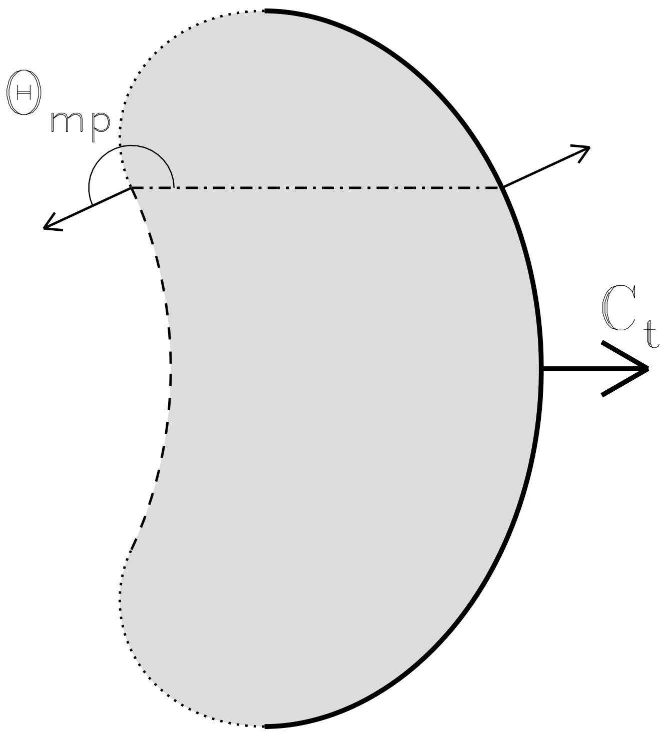
$$\frac{dY}{d\Theta} = -\frac{\cos(\Theta)}{C_t \cos(\Theta) - 1}$$

$$X = \frac{1}{C_t} \ln \frac{1}{1 - C_t \cos(\Theta)},$$

$$Y = -\frac{\Theta}{C_t} + \frac{2}{C_t\sqrt{1 - C_t^2}} \arctan \frac{(1 + C_t)\tan(\Theta/2)}{\sqrt{1 - C_t^2}}.$$

Kinematical model of a stabilized wave segment in the KL model

$$X_b = X - C_t \frac{\tau c_0^2}{D}, \quad Y_b = Y$$



$$x(\Theta_{mp}) + \tau c_t = x(\Theta_{mp} - \pi)$$

$$y(\Theta_{mp}) = y(\Theta_{mp} - \pi)$$

$$\Theta_{mp} = 2\arctan\left(-\frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} - 1}\right),$$

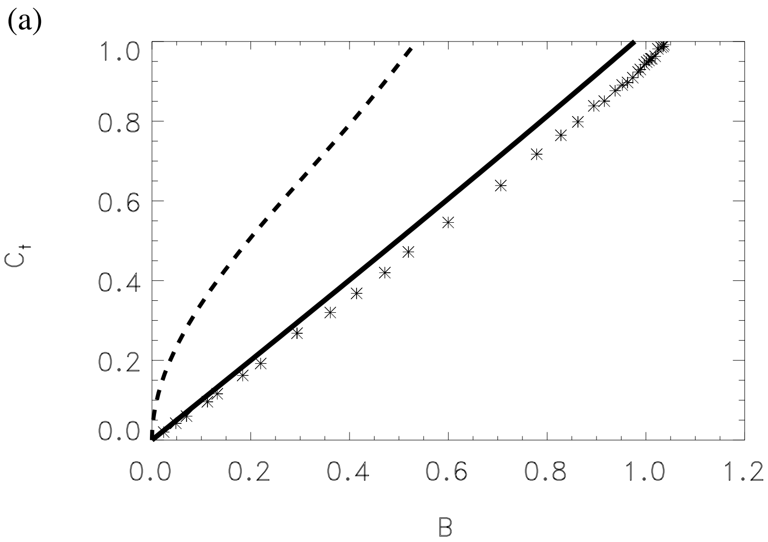
$$\mu = \left[\frac{2c_t}{\sqrt{c_0^2 - c_t^2}} \tan\left(\frac{\pi}{2c_0}\sqrt{c_0^2 - c_t^2}\right)\right]$$

$$\tau = \frac{D}{c_t^2} \ln \frac{c_0 - c_t \cos\Theta_{mp}}{c_0 + c_t \cos\Theta_{mp}}$$

$$\frac{W}{D} = \frac{2c_0}{c_t\sqrt{c_0^2 - c_t^2}} \arctan\sqrt{\frac{(c_0 + c_t)}{(c_0 - c_t)}} - \frac{\pi}{2c_t}$$

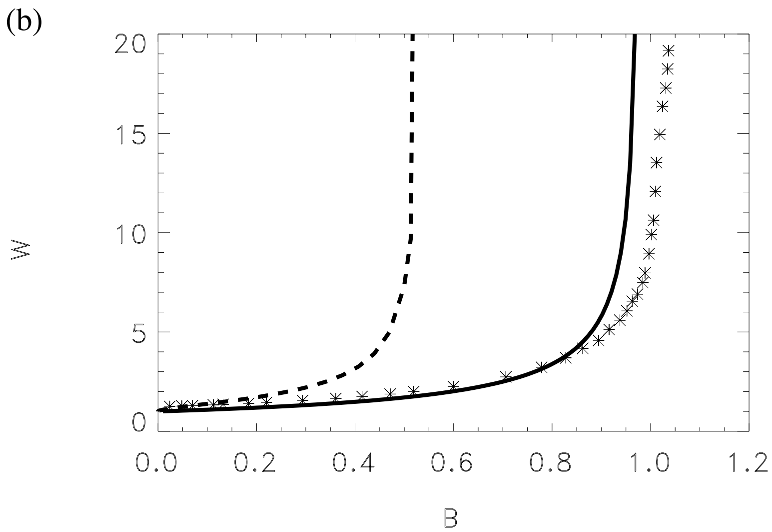
$$B = \frac{2D}{c_0^2 \tau}$$

Wave segment selection



$$C_t \cong B + 0.025\, B^3$$

$$B_{cp} = 0.977$$



$$W = \frac{2}{C_t\sqrt{1-C_t^2}} \arctan\sqrt{\frac{1+C_t}{1-C_t}} - \frac{\pi}{2C_t}$$

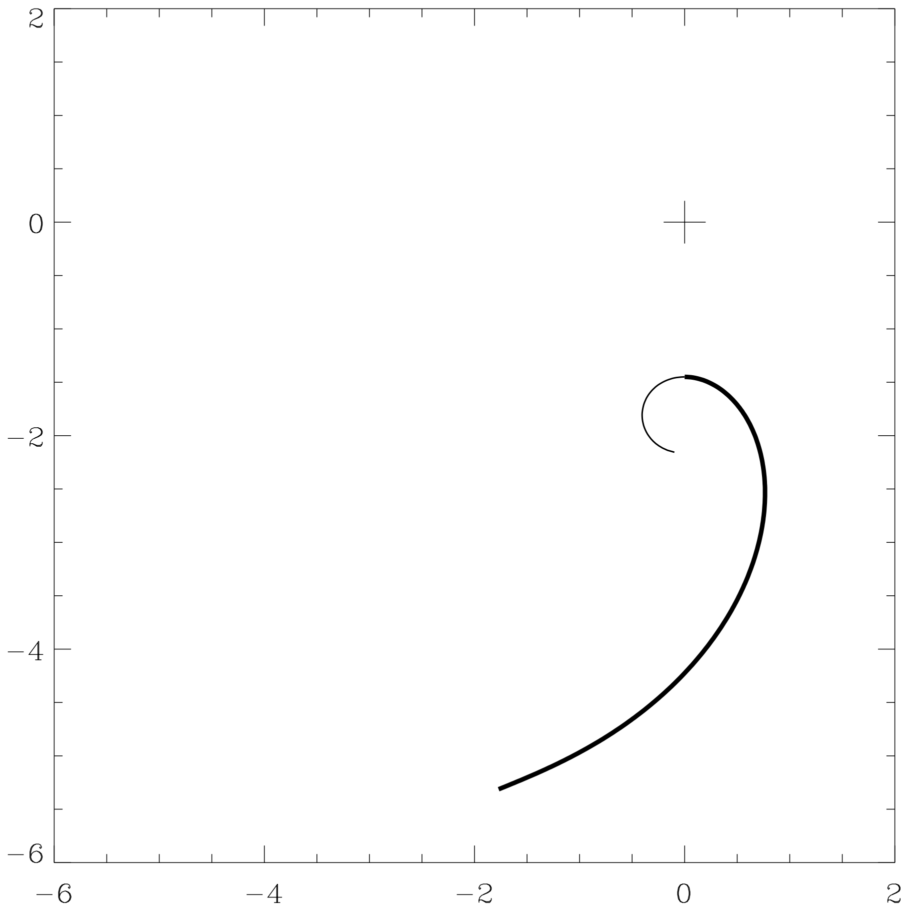
Spiral wave selection

$c_p = 2.0$

$c_t = 0.8$

$\omega = 0.552$

$R = 1.4497$



Spiral wave selection

$$c_p = 2.0$$

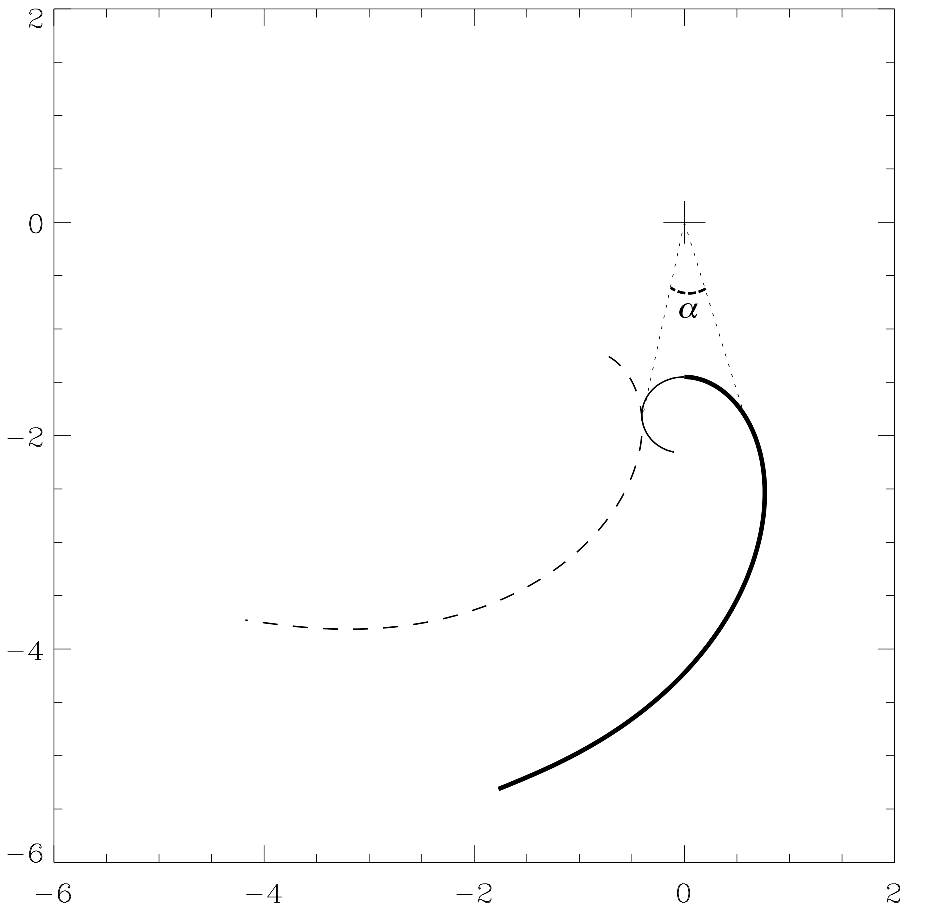
$$c_t = 0.8$$

$$\omega = 0.552$$

$$R = 1.4497$$

$$\alpha = 0.52$$

$$d_u = 0.943$$



Spiral wave selection

$c_p = 2.0$

$c_t = 0.8$

$\omega = 0.552$

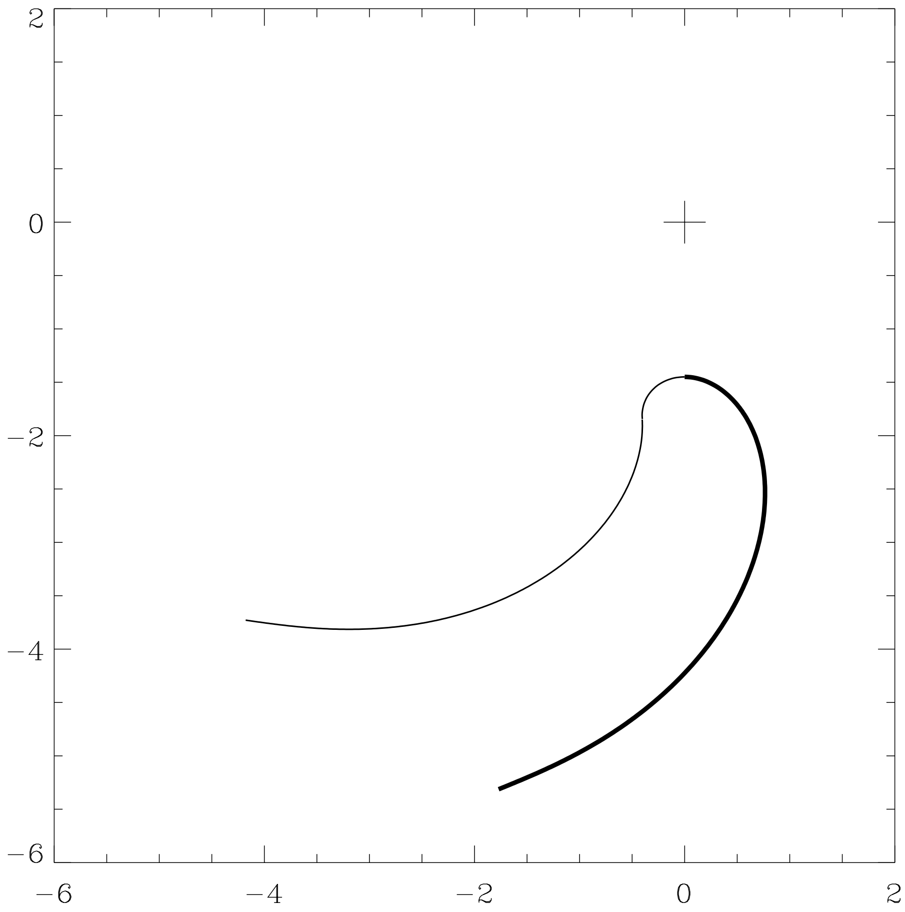
$R = 1.4497$

$\alpha = 0.52$

$d_u = 0.943$

$B = 0.53$

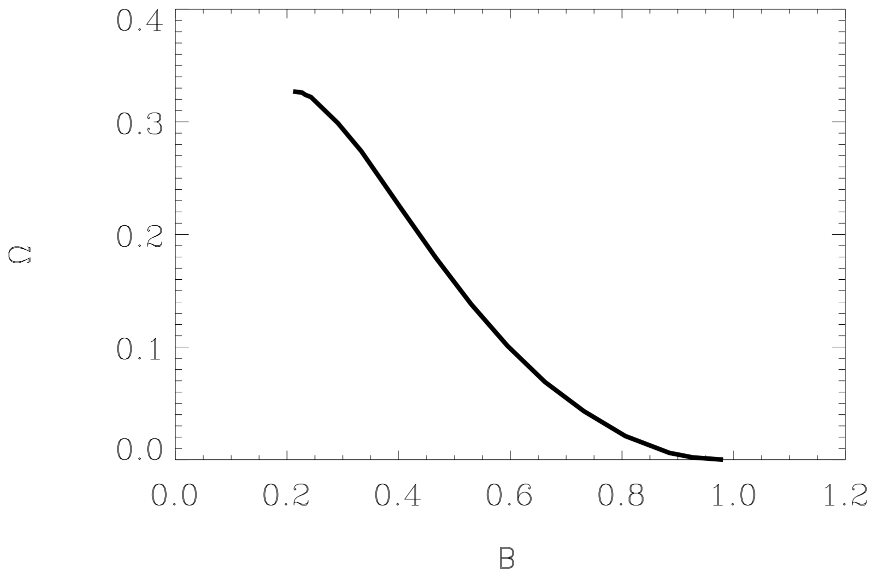
$\Omega = 0.138$

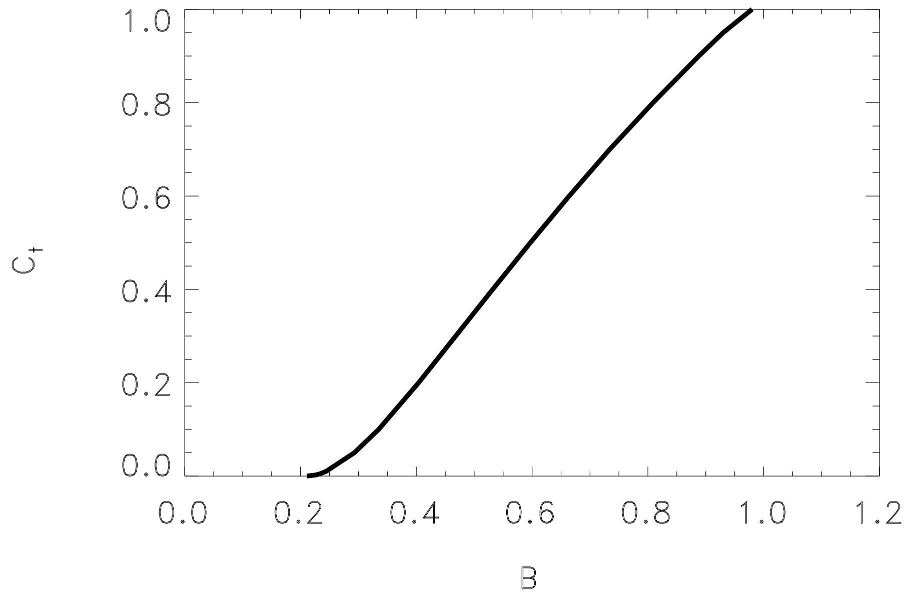


$$B = \frac{2D}{c_p^2 d_u}$$

$$\Omega = \frac{\omega D}{c_p^2}$$

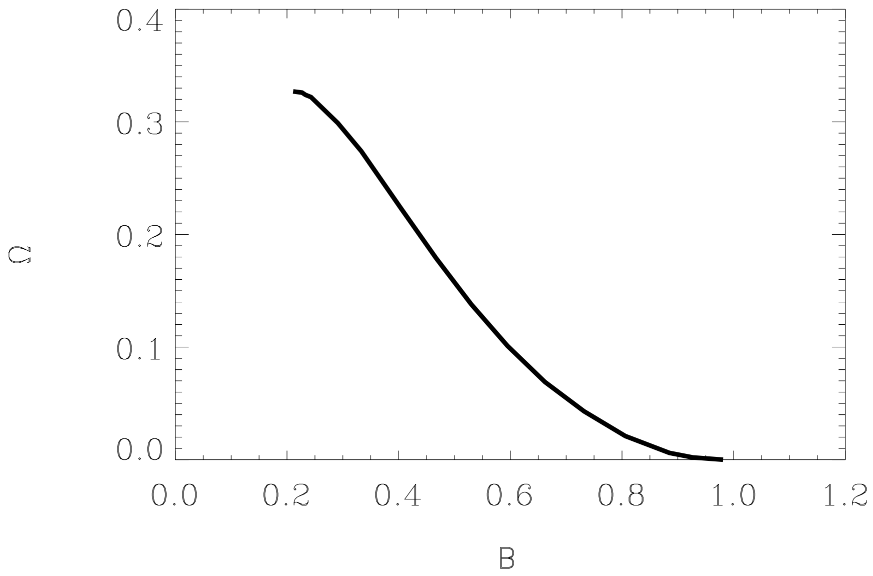
Two selected relationships

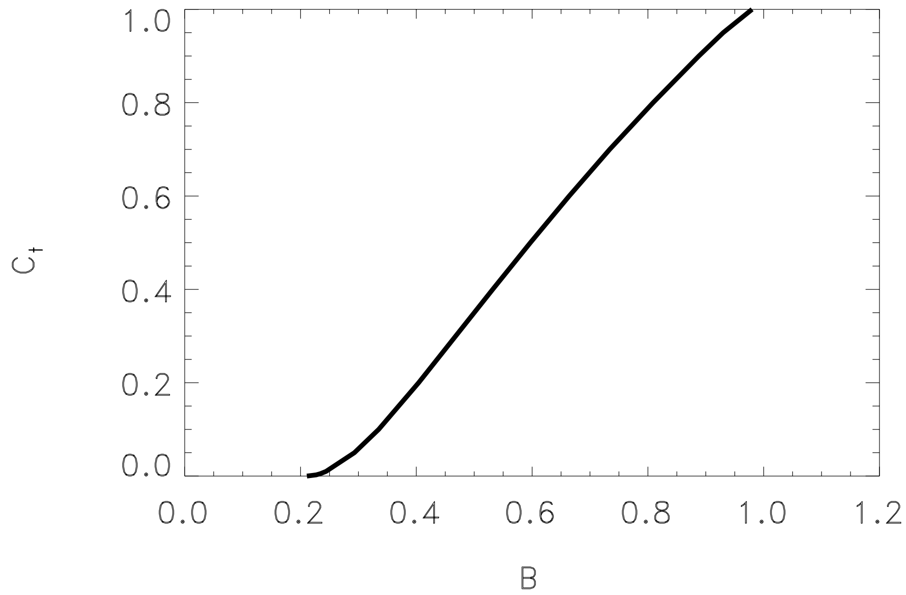




$$R_q = C_t / \Omega$$

Two selected relationships





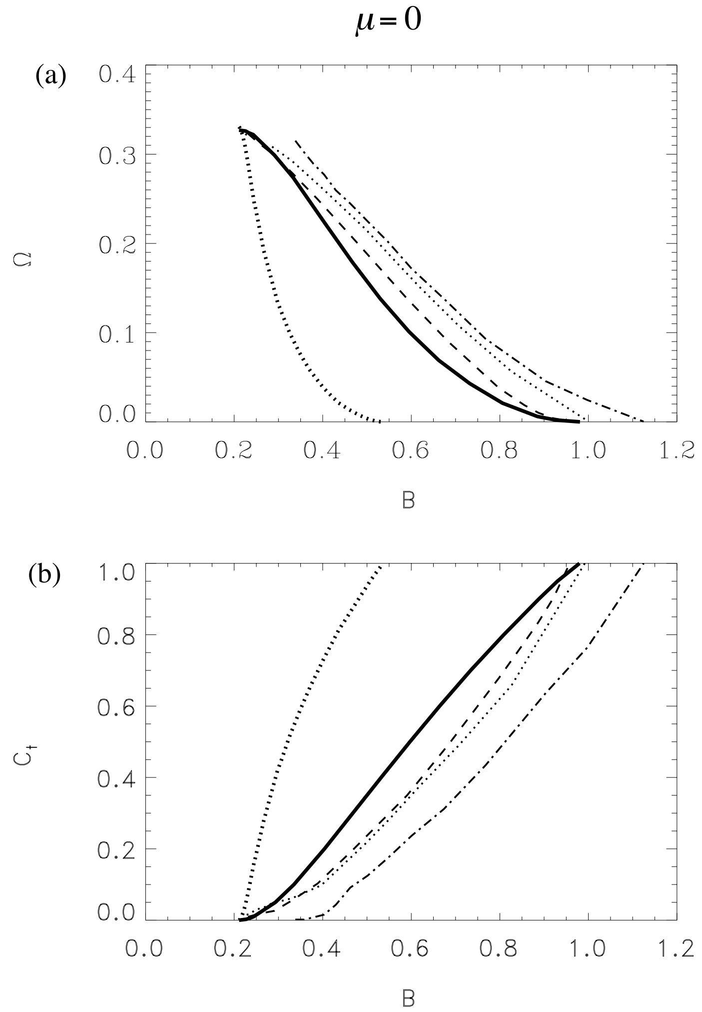
$$R_q = C_t / \Omega$$

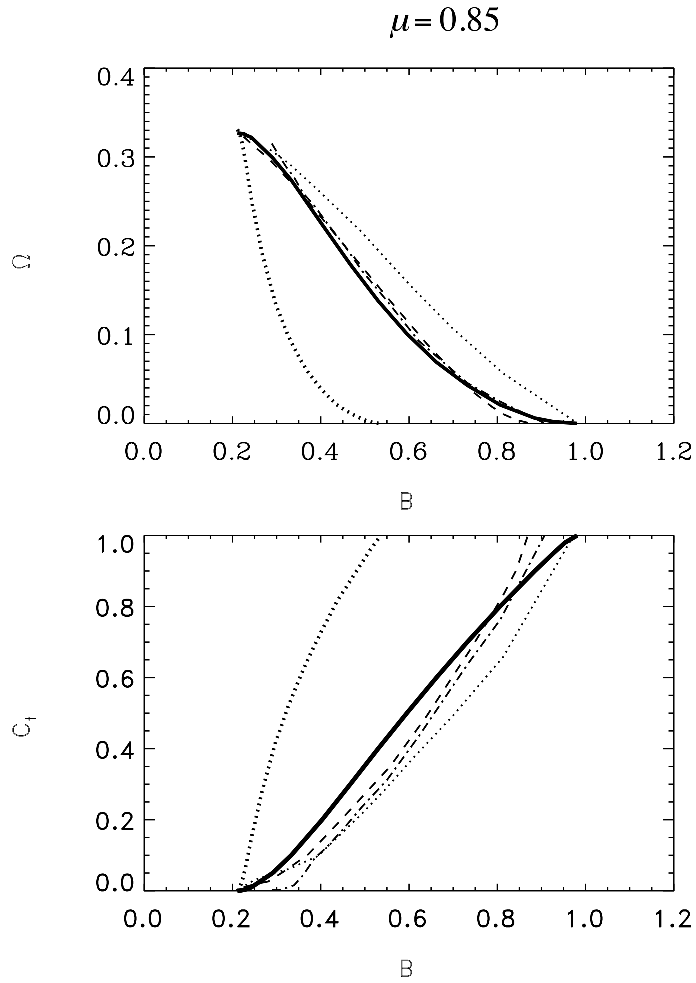
$$B_{\min} = \frac{2 * 0.331}{\pi} \approx 0.21$$

$$B_{cp} = \frac{2}{L_{cp}} = \frac{2}{2.04} \approx 0.98$$

V.S. Zykov, N. Oikawa, and E. Bodenschatz, PRL (2011)

What is a pulse duration?





$$\tau < d_u < d$$

$$B = \frac{2D}{c_p^2 d_u}$$

$$d_u = \mu\tau + (1-\mu)d$$

Summary

-Free-boundary approach reveals the selection principle, which determines spiral wave and wave segments parameters vs the medium excitability specified by the parameter B in the case of both TT and TP waves

-The critical value of the parameter B found for TP waves differs from one obtained earlier for TT waves. The minimal value of the parameter B is the same in both cases

-The results obtained in the framework of the free-boundary approach are in a good agreement with the data of the reaction-diffusion computations