# THREE-DIMENSIONAL SUPERFIELD SUPERGRAVITIES

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**We review the superfield formalisms of the three-dimensional supergravities. The $\mathcal{N}=1, d=3$ gravitational superfields are generated by the covariant spinor derivative for the tangent $GL(2,R)$ group. The basic gauge group of the $\mathcal{N}=2, d=3$ supergravity is defined in the chiral superspace.**

**The real gravitational $\mathcal{N}=2$ superfield $h^m(x,\theta,\bar{\theta})$ describes the embedding of the real gravitational superspace into the complex chiral superspace by the analogy with the Ogievetsky-Sokatchev formalism of the $\mathcal{N}=1, d=4$ supergravity. The simplest superconformal compensator is the chiral superfield.**

**The basic gauge group of the $\mathcal{N}=3, d=3$ supergravity is defined in the corresponding analytic harmonic superspace. We use the $SU(2)/U(1)$ harmonics by the analogy with the harmonic-superspace formalism of the $\mathcal{N}=2, d=4$ supergravity. The harmonic gauge superfields are defined in the decomposition of the covariant harmonic derivative. The superconformal compensator is the $\mathcal{N}=3$ analytic hypermultiplet superfield.**

## 4-DIMENSIONAL SUPERFIELD SUPERGRAVITIES

$\mathcal{N}=1, d=4$ **superfield supergravity**
**Wess, Zumino; Ogievetsky, Sokatchev; Gates, Grisaru, Roček, Siegel**

**Chirality preserving constraints for the supervielbein matrix** $E_M^A$ **were solved in terms of the** $\mathcal{N}=1$ **axial-vector gauge superfield** $H^m(x,\theta,\bar{\theta})$**. The nonlinear supergravity superdeterminant action** $\kappa^{-2}\int d^4x d^2\theta d^2\bar{\theta} E(H^m)$ **was constructed and quantized.**

$\mathcal{N}=2, d=4$ **superfield supergravity in the harmonic** $SU(2)/U(1)$ **superspace was constructed by**
**Galperin, Ivanov, Kalitzin, Ogievetsky, Sokatchev**

**The gauge supegravity superfields and the** $\mathcal{N}=2, d=4$ **matter superfields live in the Grassmann-analytic superspace. The supergravity-matter classical superfield actions were constructed in this approach, but the quantum superfield calculations were not developed in this formalism.**

**We review the superfield formalisms of the simplest three-dimensional supergravities.**

**FLAT 3-DIMENSIONAL SUPERSPACES**

$\mathcal{N}=1$ **superspace:** $z=(x^m,\theta^\mu),\quad m=0,1,2,\ \mu=1,2$

**Spinor derivatives**

$D_\mu=\partial_\mu+i\theta^\nu(\gamma^m)_{\mu\nu}\partial_m,\quad \partial_\mu\theta^\nu=\delta_\mu^\nu,\quad \partial_m x^n=\delta_m^n$

$\gamma^m\gamma^n=-\eta^{mn}I+\varepsilon^{mnp}\gamma_p,\quad \eta^{mn}=\textbf{diag}(1,-1,-1)$

$\delta_\epsilon x^m=-i(\epsilon\gamma^m\theta),\quad \delta_\epsilon\theta^\beta=\epsilon^\beta$

$\mathcal{N}=1$ **scalar superfield :** $\phi(z)$

$\mathcal{N}=1$ **Maxwell superfield :** $A_\mu(z),\quad \delta_\lambda A_\mu=D_\mu\lambda(z)$

$\mathcal{N}=2,d=3$ **superspace is analogous to the** $\mathcal{N}=1,d=4$ **superspace:**
$z=(x^m,\theta^\mu,\bar\theta^\mu),\quad m=0,1,2,\ \mu=1,2$

**Spinor derivatives**

$D_\mu=\partial_\mu+i\bar\theta^\nu(\gamma^m)_{\mu\nu}\partial_m,\quad \bar D_\mu=-\bar\partial_\mu-i\theta^\nu(\gamma^m)_{\mu\nu}\partial_m$

**Chiral** $\mathcal{N}=2$ **superspace:** $\zeta=(x_L^m,\theta^\mu),\quad x_L^m=x^m+i(\theta\gamma^m\bar\theta)$
$\mathcal{N}=2$ **scalar chiral superfield :** $\phi(x_L^m,\theta^\mu)$

**Anti-chiral superspace:** $\bar\zeta=(x_R^m,\bar\theta^\mu),\quad x_R^m=x^m-i(\theta\gamma^m\bar\theta)$

$\mathcal{N}=2,d=3$ **abelian gauge superfield:**
$\delta_\lambda V(x,\theta,\bar\theta)=i\lambda(\zeta)-i\bar\lambda(\bar\zeta),\quad W=D^\alpha\bar D_\alpha V$

$\mathcal{N}=3, d=3$ **superspace:** $z=(x^m, \theta^\mu_{(kl)}), \quad k,l=1,2$

**Automorphism group is** $SU(2)$ **and we can use the** $SU(2)/U(1)$ **harmonics** $u^\pm_k$ **and the harmonic derivatives**
$\partial^{++}u^-_k = u^+_k, \quad \partial^{--}u^+_k = u^-_k, \quad \partial^0 u^\pm_k = \pm u^\pm_k$

**Harmonic projections of spinor coordinates**
$\theta^{\mu\pm\pm} = \theta^\mu_{(kl)} u^{k\pm} u^{l\pm}, \quad \theta^{\mu 0} = \theta^\mu_{(kl)} u^{k+} u^{l-}$

**Analytic** $\mathcal{N}=3$ **superspace:** $\zeta=(x^m_A, \theta^{\mu++}, \theta^{\mu 0}, u)$
**Analytic hypermultiplet:** $q^+(\zeta)$

**Maxwell** $\mathcal{N}=3$ **superfield:** $\delta_\lambda V^{++}(\zeta) = D^{++}\lambda(\zeta)$

## $\mathcal{N}=1, d=3$ SUPERGRAVITY

**Gates, Grisaru, Roček, Siegel; Zupnik, Pak**

**Superdiffeomorphism group**

$\delta_\xi x^m = \xi^m(z), \quad \delta_\xi \theta^\mu = \xi^\mu(z)$

**Holonomic basis** $\partial_M = (\partial_m, \partial_\mu), \quad \delta_\xi \partial_M = -\partial_M \xi^N \partial_N$

**Flat-superspace basis** $D_M = (\partial_m, D_\mu)$

**The $\mathcal{N}=1$ supergravity spinor differential operator contains the superconformal gauge superfields**

$\Delta_\alpha = D_\alpha + i h^m_\alpha(z)\partial_m + h^\mu_\alpha(z) D_\mu, \quad \delta\Delta_\alpha = -\frac{1}{2}\sigma(z)\Delta_\alpha - \lambda^\beta_\alpha(z)\Delta_\beta$

**where $\sigma(z)$ is the superconformal parameter and $\lambda^\beta_\alpha(z)$ describe the $SL(2,R)$ gauge transformations.**

**Superconformal gauge condition:**

$h^\mu_\alpha(z) = 0, \quad \Delta_\alpha = D_\alpha + i h^m_\alpha \partial_m$

**The composed $SL(2,R)$ and Weyl parameters are induced by the superdiffeomorphism parameters**

$\tilde{\sigma} = \Delta_\alpha \xi^\alpha_0, \quad \tilde{\lambda}^\mu_\alpha = \Delta_\alpha \xi^\mu_0 - \frac{1}{2}\delta^\mu_\alpha \Delta_\beta \xi^\beta_0,$

$\delta\Delta_\alpha = -\Delta_\alpha \xi^\rho_0 \Delta_\rho = -(D_\alpha + i h^n_\alpha \partial_n)\xi^\rho_0 \Delta_\rho$

**The gauge transformations in this gauge are nonlinear in the basic superfields**

$\delta h^m_\alpha = -i D_\alpha \xi^m_0 - 2(\gamma^m)_{\alpha\beta}\xi^\beta_0 + h^n_\alpha \partial_n \xi^m_0 - [(D_\alpha + i h^n_\alpha \partial_n)\xi^\beta_0] h^m_\beta$

**where** $\xi^m_0 = \xi^m - i\xi^\mu \theta^\nu (\gamma^m)_{\mu\nu}, \quad \xi^\mu_0 = \xi^\mu$

**Scalar compensator of the $\mathcal{N}=1$ Poincaré supergravity**

$\Sigma(z) = 1 + \kappa\Phi(z),\ \kappa$ **is the gravitational constant**

$\delta\Phi(z) = \frac{1}{2}\Delta_\alpha \xi^\alpha_0 \left[\kappa^{-1} + \Phi(z)\right],$

$\mathcal{D}_\alpha = \Sigma\Delta_\alpha = G^M_\alpha D_M, \quad \delta\mathcal{D}_\alpha = -\tilde{\lambda}^\beta_\alpha \mathcal{D}_\beta$

**We introduce the $SL(2,R)$ spinor connection $\Omega^\rho_{\alpha,\beta}$**
$\delta\Omega^\rho_{\alpha,\beta} = -\mathcal{D}_\alpha\tilde{\lambda}^\rho_\alpha + (\lambda\Omega^\rho_{\alpha,\beta})$

**The covariant vector covariant operator is**
$\mathcal{D}_a = -\frac{i}{4}[\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} - (\Omega^\rho_{\alpha,\beta} + \Omega^\rho_{\beta,\alpha})\mathcal{D}_\rho] = G^M_a D_M$
**where $G^M_a, G^M_\alpha$ are the composed supervielbein matrix.**

**The inverse supervielbein matrix satisfies the relations**
$E^A_M G^M_B = \delta^A_B, \quad E = \mathbf{Ber} E^A_M$
$\delta E = -(\partial_m \xi^m - \partial_\mu \xi^\mu)E, \quad \delta(d^5 z E) = 0$

**The composed vector superfield connection has the form**

$$\Gamma^\rho_{a,\pi} = -\frac{i}{2}(\gamma_a)^{\alpha\beta}\{\Delta_\alpha\Omega^\rho_{\beta,\pi} - \Omega^\varphi_{\alpha,\beta}\Omega^\rho_{\varphi,\pi} - \Omega^\varphi_{\alpha,\pi}\Omega^\rho_{\beta,\varphi}\}$$

**The corresponding covariant derivative is**
$\nabla_a\mathcal{D}_\pi = \mathcal{D}_a\mathcal{D}_\pi - \Gamma^\rho_{a,\pi}\mathcal{D}_\rho, \quad \nabla_\pi\mathcal{D}_a = \mathcal{D}_\pi\mathcal{D}_a - (\gamma_a)_{\rho\xi}(\gamma^b)^{\varphi\rho}\Omega^\rho_{\pi,\varphi}\mathcal{D}_b$

**The covariant supergravity constraints have the form**
$\nabla_\alpha\mathcal{D}_\beta + \nabla_\beta\mathcal{D}_\alpha = 2i(\gamma^a)_{\alpha\beta}\mathcal{D}_a$
$\nabla_a\mathcal{D}_\pi - \nabla_\pi\mathcal{D}_a = R(\gamma_a)^\rho_\pi\mathcal{D}_\rho$,
**where $R(z)$ is the basic scalar superfield.**

**The simplest $\mathcal{N} = 1$ supergravity action is**
$\frac{1}{\kappa}\int dz^5 E(z)R(z)$

## $\mathcal{N}=2, d=3$ SUPERGRAVITY

**$\mathcal{N}=2$ supergravity formalism is based on the three-dimensional version of the Ogievetsky-Sokatchev approach. We consider the superdiffeomorphism group in the $\mathcal{N}=2, d=3$ chiral superspace**

$$\delta x^m_L = \lambda^m(\zeta), \quad \delta\theta^\mu = \lambda^\mu(\zeta), \quad \delta\bar\partial_\mu = -(\bar\partial_\mu\bar\lambda^\nu)\bar\partial_\nu$$

**This conformal $\mathcal{N}=2$ supergravity transformations preserve chirality.**

**In the anti-chiral basis we have analogous conjugated relations**

$$\delta x^m_R = \bar\lambda^m(\bar\zeta), \quad \delta\bar\theta^\mu = \bar\lambda^\mu(\bar\zeta), \quad \delta\partial_\mu = -(\partial_\mu\lambda^\nu)\partial_\nu$$

**The real supergravity superspace $Z^M = (x^m, \theta^\mu, \bar\theta^\mu)$ is embedded into the chiral superspace**

$$x^m_L = x^m + iH^m(x,\theta,\bar\theta), \quad x^m_R = x^m - iH^m(x,\theta,\bar\theta)$$

**where $H^m(x,\theta,\bar\theta) = (\theta\gamma^m\bar\theta) + h^m(x,\theta,\bar\theta)$ is the gravitational axial-vector superfield.**

**The passive gauge transformations of the $\mathcal{N}=2$ conformal supergravity have the form**

$$\delta x^m = \tfrac{1}{2}\lambda^m(x+iH,\theta) + \tfrac{1}{2}\bar\lambda^m(x-iH,\bar\theta)$$
$$\delta\theta^\mu = \lambda^\mu(x+iH,\theta), \quad \delta\bar\theta^\mu = \bar\lambda^\mu(x-iH,\bar\theta)$$
$$\delta H^m = \tfrac{i}{2}\bar\lambda^m(x-iH,\bar\theta) - \tfrac{i}{2}\lambda^m(x+iH,\theta)$$

**We consider the flat chiral basis**

$$x^m_{0L} = x^m + i(\theta\gamma^m\bar\theta), \quad \zeta_0 = (x^m_{0L}, \theta^\mu)$$
$$\lambda^m(x+iH,\theta) = T(ih)\lambda^m(\zeta_0), \quad \lambda^\mu(x+iH,\theta) = T(ih)\lambda^\mu(\zeta_0),$$
$$T(ih) = 1 + ih^m\partial_m - \tfrac{1}{2}h^mh^n\partial_m\partial_n - \tfrac{i}{6}h^mh^nh^r\partial_m\partial_n\partial_r + \ldots$$

**The covariant spinor derivatives in the central coordinates** $x^m, \theta^\mu, \bar{\theta}^\mu$ **have the form**

$\Delta_\alpha = D_\alpha + i\Delta_\alpha h^m \partial_m = D_\alpha + iD_\alpha h^n [(I - i\partial h)^{-1}]_n^m \partial_m,$

$\bar{\Delta}_\alpha = \bar{D}_\alpha - i\bar{\Delta}_\alpha h^m \partial_m = \bar{D}_\alpha - i\bar{D}_\alpha h^n [(I + i\partial h)^{-1}]_n^m \partial_m$

**The tangent** $GL(2, C)$ **transformations of these derivatives are induced by the chiral diffeomorphism transformations**

$\delta\Delta_\alpha = -(\Delta_\alpha \lambda^\beta)\Delta_\beta, \quad \delta\bar{\Delta}_\alpha = (\bar{\Delta}_\alpha \bar{\lambda}^\beta)\bar{\Delta}_\beta$

**The basic superfield blocks** $c^m_{\mu\nu}\partial_m$ **can be constructed via the anticommutator of spinor covariant derivatives**

$\frac{1}{2}\{\Delta_\mu, \bar{\Delta}_\nu\} = -i\{(\gamma^m)_{\mu\nu} + \frac{1}{2}[\Delta_\mu, \bar{\Delta}_\nu]h^m\}\partial_m = c^m_{\mu\nu}\partial_m$

**We consider the superconformal chiral parameter**

$j = \partial^L_m \lambda^m - \partial_\mu \lambda^\mu, \quad \delta d^3x_L d^2\theta = j d^3x_L d^2\theta = j d^5\zeta$

**and introduce the chiral superfield compensator**

$\delta\Phi = -\frac{1}{2} j\Phi, \quad \delta(d^5\zeta\Phi^2) = 0$

**This chiral compensator allow us to construct the** $\mathcal{N} = 2$ **Poincaré supergravity action.**

## $\mathcal{N}=3, d=3$ SUPERGRAVITY

**We use the analogy with the Galperin-Ivanov-Ogievetsky-Sokatchev formalism in $\mathcal{N}=2, d=4$ supergravity. Let us consider the arbitrary transformations of the $\mathcal{N}=3, d=3$ harmonic analytic superspace**

$$\delta x_A^m = \lambda^m(\zeta), \quad \delta\theta^{++\mu} = \lambda^{++\mu}(\zeta), \quad \delta\theta^{0\mu} = \lambda^{0\mu}(\zeta),$$
$$\delta u_k^+ = \lambda^{++}(\zeta) u_k^-, \quad \zeta = (x_A^m, \theta^{++\mu}, \theta^{0\mu}, u_k^\pm),$$

**where $\lambda^m(\zeta), \lambda^{++\mu}(\zeta), \lambda^{0\mu}(\zeta), \lambda^{++}(\zeta)$ are the analytic parameters of the conformal supergravity.**

**The harmonic and spinor derivatives in the flat analytic superspace are**

$$\mathcal{D}^{++} = \partial^{++} + 2i\theta^{++\alpha}\theta^{0\beta}\partial^A_{\alpha\beta} + \theta^{++\alpha}\partial^0_\alpha + 2\theta^{0\alpha}\partial^{++}_\alpha, \quad \partial^A_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial^A_m,$$
$$\mathcal{D}^{--} = \partial^{--} - 2i\theta^{--\alpha}\theta^{0\beta}\partial^A_{\alpha\beta} + \theta^{--\alpha}\partial^0_\alpha + 2\theta^{0\alpha}\partial^{--}_\alpha,$$
$$\mathcal{D}^0 = \partial^0 + 2\theta^{++\alpha}\partial^{--}_\alpha - 2\theta^{--\alpha}\partial^{++}_\alpha, \quad [\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0,$$
$$D^{++}_\alpha = \partial^{++}_\alpha, \quad D^{--}_\alpha = \partial^{--}_\alpha + 2i\theta^{--\beta}\partial^A_{\alpha\beta}, \quad D^0_\alpha = -\frac{1}{2}\partial^0_\alpha + i\theta^{0\beta}\partial^A_{\alpha\beta},$$
$$\partial^A_m x^n_A = \delta^n_m, \quad \partial^0_\alpha\theta^{0\beta} = \delta^\beta_\alpha, \quad \partial^{\mp\mp}_\alpha\theta^{\pm\pm\beta} = \delta^\beta_\alpha.$$

**The analytic integral measure is**

$$d\zeta^{-4} = \frac{1}{16}d^3x_A(\partial^{--\alpha}\partial^{--}_\alpha)(\partial^{0\alpha}\partial^0_\alpha)du, \qquad (0.1)$$
$$\delta d\zeta^{-4} = (\partial^A_m\lambda^m + \partial^{--}\lambda^{++} - \partial^{--}_\mu\lambda^{++\mu} - \partial^0_\mu\lambda^{0\mu})d\zeta^{-4} = -2\Lambda d\zeta^{-4}.$$

**Nonanalytic transformations have the form**

$$\delta\theta^{--\mu} = \Lambda^{--\mu}(\zeta, \theta^{--}).$$

**We define the Killing operator**

$$K = \lambda^m \partial_m + \lambda^{++} \partial^{--} + \lambda^{++\mu} \partial^{--}_\mu + \lambda^{0\mu} \partial^0_\mu + \Lambda^{--\mu} \partial^{++}_\mu$$
$$= \Lambda^m \partial_m + \lambda^{++} \mathcal{D}^{--} + \Lambda^{++\mu} D^{--}_\mu + \Lambda^{0\mu} D^0_\mu + \Lambda^{--\mu} D^{++}_\mu$$

**Constraints for the flat parameters:**

$$D^{++}_\mu \Lambda^m = -4i\lambda^{++}(\gamma^m)_{\mu\nu}\theta^{0\nu} + 2i\lambda^{++\nu}(\gamma^m)_{\mu\nu} = 2i(\gamma^m)_{\mu\nu}\Lambda^{++\nu},$$
$$D^{++}_\nu \Lambda^{0\mu} = -2\delta^\mu_\nu \lambda^{++}.$$

**The basic operator of the $\mathcal{N} = 3$ conformal supergravity contains the gauge gravitational superfields**
$\Delta^{++} = \mathcal{D}^{++} + G^{++} + h^\mu D^{++}_\mu, \quad [G^{++}, D^{++}_\nu = 0,$
$G^{++} == h^{++m}\partial_m + h^{(+4)}\mathcal{D}^{--} + h^{(+4)\mu} D^{--}_\mu + h^{++\mu} D^0_\mu$

**The transformation of the basic analytic harmonic operator defines transformations of the gravitational superfields**
$\delta\Delta^{++} = -\lambda^{++}\mathcal{D}^0$

**We can also construct the nonanalytic harmonic operator** $\Delta^{--}, \quad \delta\Delta^{--} = -(\Delta^{--}\lambda^{++})\Delta^{--}$ **satisfying the constraints**
$[(\Delta^{++} - h^{(+4)}\Delta^{--}), \Delta^{--}] = \mathcal{D}^0$

**We define the covariant spinor and vector derivatives**
$\Delta^0_\alpha = \frac{1}{2}[\Delta^{--}, D^{++}_\alpha], \quad \Delta^{--}_\alpha = [\Delta^{--}, \Delta^0_\alpha], \quad \Delta_a = \frac{i}{2}(\gamma_a)^{\alpha\beta}\{\Delta^0_\alpha, \Delta^0_\beta\}$

**Superconformal compensator hypermultiplet**
$\delta q^+(\zeta) = \Lambda q^+(\zeta), \quad \Lambda = -\frac{1}{2}(\partial^A_m \lambda^m + \partial^{--}\lambda^{++} - \partial^{--}_\mu \lambda^{++\mu} - \partial^0_\mu \lambda^{0\mu})$