Classical self-energy of charges near higher-dimensional black holes and anomalies

Andrei Zelnikov
and
Valeri Frolov

Ginzburg Conference 2012

May 29, 2012

The problem of the electromagnetic origin of the electron mass has a long history. It first was formulated in the classical theory when in 1881 Thompson demonstrated that the self-energy of the electromagnetic field contributes to the inertial mass of a charged particle.

This idea was then elaborated in the works by Lorentz [1989], Abraham [1903], Poincaré [1905], Fermi [1921] and others.

- For a simple model of a uniformly charged sphere of radius ε the electrostatic energy is $E=e^2/\varepsilon$. However it was shown by Abraham [1904-1905] the relation between energy and momentum for such a particle differs from the standard one by a factor 4/3. This factor disappears if one includes in the definition of the self-energy a contribution of additional (non-electromagnetic) forces that are required to make the system stable. To solve 4/3-problem Poincaré [1906] introduced special sort of non-electromagnetic pressure.

- In quantum electrodynamics the self-energy of an electron diverges and, hence, should be regularized and renormalized. A classical self-energy of pointlike charges suffers similar divergences. Quantum field theory provides us with methods to deal with this problem systematically.

  Classical self-energy of an electron can be derived as the limit of its quantum value [Vilenkin and Fomin, Efimov]

- We apply QFT methods to resolve the problems with ambiguities and model dependence of the classical self-energy of charged particles.

- In higher dimensions these problems are much more serious than in four dimensions, and new features appear.

- We consider static scalar charges in the gravitational field of higher dimensional black holes.
- In this case radiation-reaction effects vanish.

We will show that unexpected contributions to the self-energy $E = m\sqrt{|g_{00}|}$ and self-force appear in odd-dimensional spacetimes. This effect is classical but is closely related to quantum conformal anomaly.

$$\Delta m = -\frac{q^2}{2} G_{\mathrm{reg}}(x,x) = -\frac{q^2}{2} < \varphi^2 >_{\mathrm{ren}}$$

$$\Delta m = -\frac{1}{2} \boxed{\frac{q^2}{\hbar c}} < \varphi^2 >_{\mathrm{(quantum)\ ren}}$$

φ **is the field defined on (D-1)-dimensional spatial slice (t=const) of a static spacetime**

$$< \varphi^2 >_{\mathrm{ren}} \sim 1$$

$$< \varphi^2 >_{\mathrm{(quantum)\ ren}} \sim \hbar$$

Anomaly of what?

Formally the functional of the self-energy $E = m\sqrt{|g_{00}|}$ of a charge distribution is invariant under transformations of the static metric.

$$g_{ab} = \Omega^2(x)\,\bar{g}_{ab}, \qquad g_{00} = \Omega^{-(D-3)}(x)\,\bar{g}_{00}$$

But E diverges for point-like sources. Regularization breaks this invariance and acquires an anomalous contribution

$$\Delta m = -\frac{q^2}{2} < \varphi^2 >_{\text{ren}}$$

$$\boxed{g^{\frac{D-3}{2(D-1)}}\left(< \varphi^2 >_{\text{ren}} + A\right) = \text{const}}$$

$$g = \det g_{ab}$$

$$\sim \Omega^{(D-3)}$$

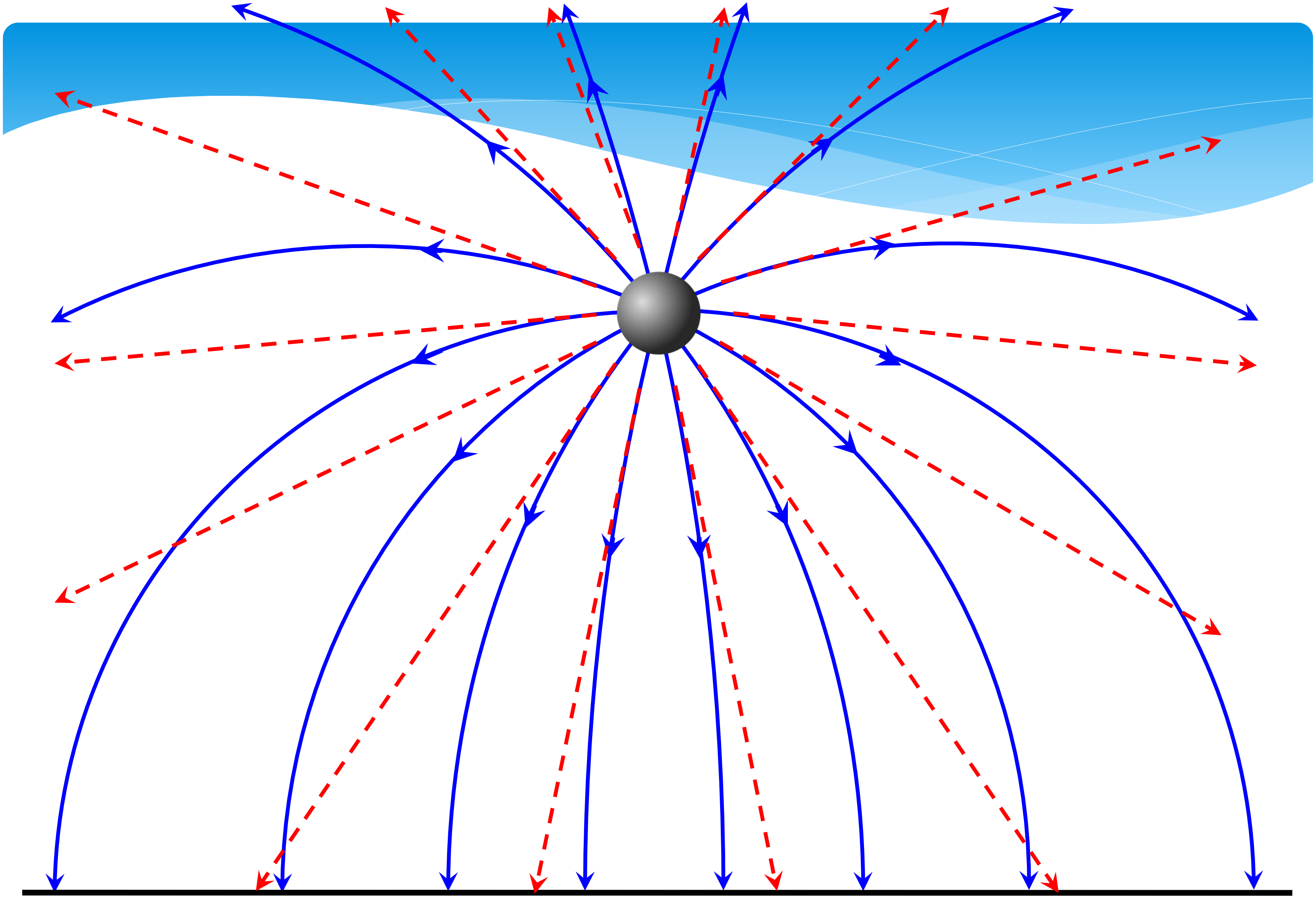
similar to the conformal anomaly in QFT.

In some simple cases, like the charge in a homogeneous gravitational field, the self-energy can be calculated exactly and we can test our approach.

In 4D the energy of an electric charge in a homogeneous gravitational field is reduced by an amount proportional to its acceleration.

$$E^{em} = \frac{e^2}{2\varepsilon} - \frac{e^2 w}{2}$$

UV – cutoff radius



Horizon

For calculations of the self-energy of static charges one has to know only **static Green functions.**

Fortunately, in some interesting cases: static charges near **4-dimensional** Schwarcschild or Reissner-Nordström black holes the static Green functions are known exactly
[Copson (1928), Leaute-Linet (1976); Linet (1976)].

As the result one can show that electron gets an additional positive energy due to the self-interaction [Smith Will (1980); Frolov and Zelnikov (1980,1981); Ritus 1981, Lohiya (1982)].

$$E^{em} = \underbrace{\left(m_{bare} + \frac{e^2}{2\varepsilon} \right)}_{m_{ren}} | g_{00} |^{1/2} + \boxed{\frac{e^2 M}{2\, r^2}}$$

which leads to an additional repulsive (from the black hole) self-force.

For a scalar charge q near a four-dimensional Reissner-Nordström black hole the self-force vanishes

$$E^{sc} = \underbrace{\left(m_{bare} + \frac{q^2}{2\varepsilon}\right)}_{m_{ren}} | g_{00} |^{1/2} + \boxed{0}$$

- This is IR effect and it is quite subtle. One has to be very delicate in dealing with the model of a classical electron because a minute discrepancy in calculations of the UV-divergent classical energy of the electron in the external gravitational field may easily overshoot the effect itself.

- In higher dimensions the UV-divergencies are much stronger and have richer structure than in four dimensions and one has to be infinitely more accurate in describing the model.

- **Knowledge of the exact higher-dimensional Green functions is a plus for treatment of IR behavior of fields generated by charges in curved spacetime.**

- Some additional finite terms may also survive **after renormalizations of UV-divergencies.**

Self-energy of a scalar charge in a static spacetime

Minimally coupled massless scalar field $\Box\,\Phi = -4\pi J$

$$I = -\frac{1}{8\pi}\int d^D y\sqrt{-\mathrm{g}}\ \Phi^{;\alpha}\Phi_{;\alpha} + \int d^D y\sqrt{-\mathrm{g}}\ J\,\Phi$$

$$\mathrm{g} = \det \mathrm{g}_{\mu\nu} = -\alpha^2 g\,, \qquad g = \det g_{ab}\,.$$

In a static D-dimensional spacetime with the metric:

$$ds^2 = -\alpha^2 dt^2 + g_{ab}\,dx^a dx^b$$

This scalar field is not conformal.

The energy E of a static configuration of fields is

$$E = \frac{1}{8\pi}\int d^{D-1}x\sqrt{g}\,\alpha\;\Phi^{;a}\Phi_{;a}$$

Introduce a new field variable $\Phi = \alpha^{-1/2}\varphi$

$$E = \frac{1}{8\pi}\int d^{D-1}x\sqrt{g}\,g^{ab}\left(\varphi_{,a} - \frac{\alpha_{,a}}{2\alpha}\varphi\right)\left(\varphi_{,b} - \frac{\alpha_{,b}}{2\alpha}\varphi\right)$$

$$F\varphi \equiv (\Delta + V)\varphi = 0$$

(D-1)-dimensional field theory

$$\Delta \equiv g^{ab}\nabla_a\nabla_b$$

$$V = \frac{(\nabla\alpha)^2}{4\alpha^2} - \frac{\Delta\alpha}{2\alpha}$$

- The energy E is a functional of (D-1)-dimensional metric g_{ab}, \`dilaton\` field α , and the scalar field φ .

  This functional formally looks like the functional of (D-1)-dimensional Euclidean action.

- Consider continuous transformations of E described by a function $\Omega(x)$:

$$n = D - 3$$

$$g_{ab} = \Omega^2 \bar{g}_{ab}, \qquad \alpha = \Omega^{-n} \bar{\alpha}, \qquad \varphi = \Omega^{-n/2} \bar{\varphi}$$

- The functional E is invariant under these transformations.

The operator F transforms homogeneously

$$F = \Omega^{-2-\frac{n}{2}} \bar{F} \ \Omega^{\frac{n}{2}}$$

Self-energy of a scalar charge q

$$F\varphi = -4\pi j$$

$$j = \alpha^{1/2} J$$

Pointlike source

$$J(x) = q\int_{-\infty}^{\infty} d\tau\, \delta^{D-1}(x,x')\,\frac{\delta(t-t'(\tau))}{\alpha(x)} = q\,\delta^{D-1}(x,x')$$

$$\delta^{D-1}(x,x') = \frac{\delta^{D-1}(x-x')}{\sqrt{g}}$$

$$E = \int_{\Sigma} T_{\mu\nu}\xi^{\mu} d\Sigma^{\nu} = -\frac{1}{2}\int_{\Sigma} \varphi\, j\sqrt{g}\; d^{D-1}x$$

$$= -\frac{1}{2}\int_{\Sigma}\int_{\Sigma'} j\, G\; j'\sqrt{g}\; d^{D-1}x\sqrt{g'}\; d^{D-1}x'$$

- **Thus**

$$\boxed{E = -\frac{q^{2}}{2}\alpha(x)\, G(x,x)}$$

- As expected, the obtained expression for the self-energy of a pointlike charge is **divergent**. To deal with this problem we shall use the point-splitting method, similar to the regularization schemes adopted in the quantum field theory

$$G(x,x) \;\rightarrow\; G_{\text{reg}}(x,x) = \lim_{x\to x'}\left[G(x,x') - G_{\text{div}}(x,x')\right]$$

$$E = -m\,u_{\mu}\xi^{\mu} = m\,\alpha(x).$$

- $$\boxed{\Delta\, m = -\frac{q^2}{2} G_{\text{reg}}(x,x) = -\frac{q^2}{2} < \varphi^2 >_{\text{ren}}}$$

The Green function $G(x, x')$ $n = D - 3$

$$F\,G(x,x') = -\delta^{n+2}(x,x')$$

transforms as $G(x,x') = \Omega^{-\frac{n}{2}}(x)\;\bar{G}(x,x')\;\Omega^{-\frac{n}{2}}(x')$

Therefore $g^{\frac{n}{2(n+2)}} < \varphi^2 >$ is invariant under the transformations.

But $G_{\mathrm{div}}(x,x')$ does not respect this invariance and, hence,

$$g^{\frac{n}{2(n+2)}} < \varphi^2 >_{ren} \neq \mathrm{const}$$

- However, one can find such anomalous term $A(x)$ that

$$g^{\frac{n}{2(n+2)}} \left(< \varphi^2 >_{\mathrm{ren}} + A \right) = \mathrm{const}$$

Divergent terms

$$G_{\rm div}(x,x') = \Delta^{1/2}(x,x')\frac{1}{(2\pi)^{\frac{n}{2}+1}}\sum_{k=0}^{[n/2]}\frac{\Gamma\left(\frac{n}{2}-k\right)}{2^{k+1}\sigma^{\frac{n}{2}-k}}a_k(x,x')$$

$$\frac{\Gamma\left(\frac{n}{2}-k\right)}{2^{k+1}\sigma^{\frac{n}{2}-k}}a_k(x,x')\,|_{k=n/2}\rightarrow -\frac{\ln\sigma(x,x')+\gamma-\ln 2}{2^{\frac{n}{2}+1}}a_{n/2}(x,x').$$

$$g_{ab} = \Omega^2\,\bar{g}_{ab}, \qquad g^{ab} = \Omega^{-2}\,\bar{g}^{ab},$$

$$g^{ab}\sigma_a\sigma_b = 2\sigma, \qquad \bar{g}^{ab}\bar{\sigma}_a\bar{\sigma}_b = 2\bar{\sigma}.$$

World function

$$\sigma = \bar{\sigma}\,\Omega(\mathbf{x})\Omega(\mathbf{x}')\,[\,1+\frac{1}{12\Omega^2}(-2\Omega\Omega_{:ab}+4\Omega_{:a}\Omega_{;b}-\Omega_{:c}\Omega^{:c}\,\bar{g}_{ab})\bar{\sigma}^a\bar{\sigma}^b]+O(\sigma^{5/2}).$$

Schwinger-DeWitt coefficients

$$a_0(x,x') = 1, \qquad a_k(x,x') = \ldots$$

Van Vleck-Morette determinant

$$\begin{aligned}\Delta^{1/2} &= 1 + \frac{1}{12} R_{ab} \sigma^a \sigma^b + O(\sigma^{3/2}) \\ &= 1 + \frac{1}{12} \bar{R}_{ab} \bar{\sigma}^a \bar{\sigma}^b \\ &+ \frac{1}{12\Omega^2} [-n\Omega\Omega_{:ab} + 2n\Omega_{:a}\Omega_{:b} - \left(\Omega\Omega^{:c}_{:c} + (n-1)\Omega^{:c}\Omega_{:c}\right) \bar{g}_{ab}] \bar{\sigma}^a \bar{\sigma}^b + O(\sigma^{3/2}).\end{aligned}$$

- $$<\varphi^2>_{\text{ren}} = \Omega^{-n} <\bar{\varphi}^2>_{\text{ren}} - B$$

$$B(x) = \lim_{x' \to x} \left[G_{\text{div}}(x,x') - \frac{\bar{G}_{\text{div}}(x,x')}{\Omega^{n/2}(x)\Omega^{n/2}(x')} \right].$$

- **In 4D (and any even dimension) the anomaly *B=0***
- **In 5D** $B = -\frac{1}{48\pi^2}\Omega^{-3}\Omega^{:c}_{:c} - \frac{\bar{a}_1}{8\pi^2}\Omega^{-2}\ln(\Omega)$

$$R = \frac{1}{\Omega^2}\left(\bar{R} - 6\Omega^{-1}\Omega^{:c}_{:c}\right),$$

$$a_1 = \frac{1}{6}R + V = \frac{1}{\Omega^2}\left(\frac{1}{6}\bar{R} + \bar{V}\right) = \frac{1}{\Omega^2}\bar{a}_1.$$

$$V = \frac{1}{\Omega^2}\left(\bar{V} + \Omega^{-1}\Omega^{:c}_{:c}\right),$$

The anomaly B can be obtained from the transformation law

- $$g^{\frac{n}{2(n+2)}}\left(<\varphi^2>_{\mathrm{ren}}+A\right)=\mathrm{const}$$

Where A depends on g_{ab} and α

In 5D:

- $$A(x)=\frac{1}{288\pi^2}R-\frac{1}{64\pi^2}\ln(g)\,a_1(x)$$
  $$a_1(x)=\frac{1}{6}R+V$$

If one knows $<\bar{\varphi}^2>_{\text{ren}}$ in some reference spacetime, Then using this anomaly one can derive $<\varphi^2>_{\text{ren}}$ in all other spacetimes related to the reference one by transformations we have discussed.

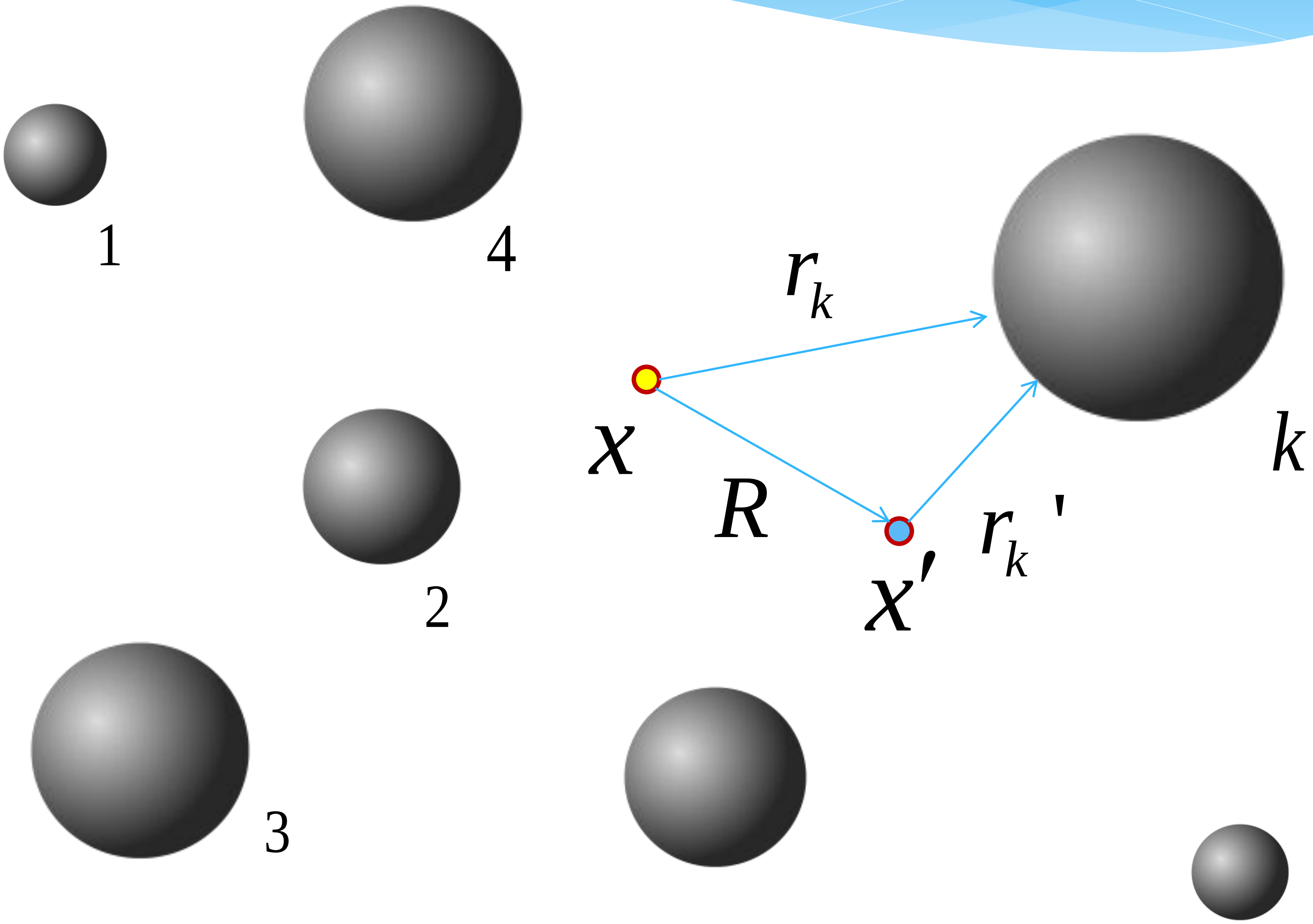
$$<\varphi^2>_{\text{ren}} = \Omega^{-n} <\bar{\varphi}^2>_{\text{ren}} - B$$

Higher dimensional Majumdar-Papapetrou metrics

$$ds^2 = -U^{-2}\, dt^2 + U^{2/n} \delta_{ab}\, dx^a dx^b$$

$$U = 1 + \sum_k \frac{M_k}{r_k^n}, \qquad r_k = \sqrt{\delta_{ab}(x^a - x_k^a)(x^b - x_k^b)}$$

Where x_k^a is the spatial position of the k-th extremal black hole


1
4
r_k
x
R
x'
r_k'
k
2
3

One can see that the transformation

$$\Omega(x) = U^{1/n}(x)$$

connects the Majumdar-Papapetrou metric to the Minkowski D-dimensional metric and

$< \varphi^2 >_{\text{ren}} = -B$ **because** $< \bar{\varphi}^2 >_{\text{ren}} = 0$

In 4D $\Delta\, m = 0$

In 5D

$$\boxed{\Delta\, m = \frac{q^2}{576\pi^2} R}$$

Where R **is the Ricci scalar of the spatial metric** g_{ab}

Summary

The self-energy of static scalar sources of a minimally coupled massless scalar field is invariant under special symmetry transformations. This exact transformation law makes possible to relate the self-energy of a charge in the physical spacetime to the self-energy in some reference spacetime, where its calculation may be significantly simpler.

In the case of Majumdar-Papapetrou spacetimes it happens that this symmetry relates Majumdar-Papapetrou spacetimes to the flat Minkowski spacetime.

Regularization procedure breaks this symmetry and results in appearance of the anomaly. We have presented an approach to study the self-energy of pointlike charges based on calculation of the self-energy anomaly.

$$\Delta m = -\frac{q^2}{2} < \varphi^2 >_{\mathrm{ren}}$$

$$< \varphi^2 >_{\mathrm{ren}} = \Omega^{-2} < \bar{\varphi}^2 >_{\mathrm{ren}} - B$$

In even dimensions $B = 0$

In odd dimensions $B \neq 0$ **in a generic case**

Maxwell field

$$F^{\mu\varepsilon}{}_{;\varepsilon} = 4\pi J^{\mu}$$

$$\delta^{ab}\,\partial_a\left(U^2\partial_b A_0\right) = +4\pi U^{2/n}\,J^0$$

$$A_0 = -U^{-1}\psi$$

$$\left[\Delta-(U^{-1}\Delta U)\right]\psi = -4\pi U^{-1+\frac{2}{n}}\,J^0$$

$$\uparrow$$

$$\Delta U \sim \sum_k M_k\,\delta^{n+2}(\mathbf{x}-\mathbf{x}_k)$$

$$\mathcal{G}_{00}(\mathbf{x},\mathbf{x}') = -\frac{\Gamma\left(\frac{n}{2}\right)}{4\pi^{1+\frac{n}{2}}}\cdot\frac{1}{U(x)U(x')}\left[\frac{1}{R^n} + \boxed{\sum_k \frac{M_k}{r_k^n\, r'^{\,n}_k}}\,\right]$$

$$R^2 = \delta_{ab}(x^a - x'^a)(x^b - x'^b)$$

$$r_k^2 = \delta_{ab}(x^a - x_k^a)(x^b - x_k^b)$$

$$U(x) = 1 + \sum_k \frac{M_k}{r_k^n}$$

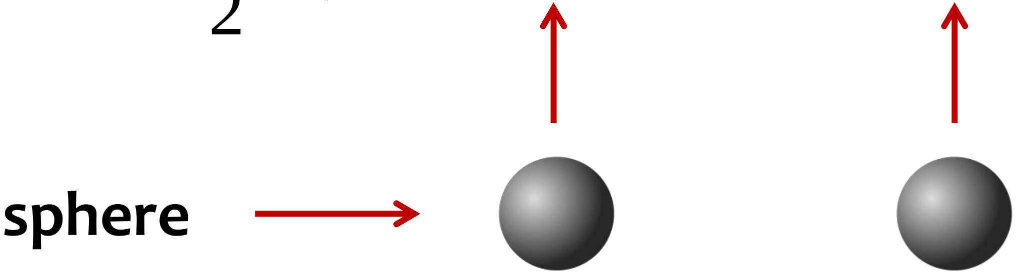
Self-energy of charges

$$E_{self}=\int_{\Sigma}T_{\mu\nu}\xi^{\mu}d\Sigma^{\nu}$$

Scalar charge

$$E_{self}^{sc}=-\frac{1}{2}\int_{\Sigma}\varphi J\sqrt{-g}\,d^{n+2}x+\boxed{\int_{\partial\Sigma}\cdots\cdots}\longrightarrow 0$$

$$=-\frac{1}{2}\int\sqrt{-g(x)}\,J(x)\,\mathcal{G}(x,x')\,J(x')\sqrt{-g(x')}\,d^{n+2}xd^{n+2}x'$$



The other way is to consider point-like sources

$$J\sqrt{-g} = q\sqrt{-g_{00}(x)}\;\delta(x^a - x'^a) = qU^{-1}(x)\;\delta(x^a - x'^a)$$

but use the regularized Green function

$$\mathcal{G}(x, x') \rightarrow \mathcal{G}_{reg}(x, x')$$

$$E_{self}^{sc} \rightarrow -\frac{1}{2}q^2U^{-2}(x)\;\mathcal{G}_{reg}(x, x)$$

Electric charge

$$E_{self}^{em} = -\frac{1}{2}\int_{\Sigma} A_0 J^0 \sqrt{-g}\; d^{n+2}x$$

$$= -\frac{1}{2}\int \sqrt{-g(x)}\, J^0(x)\, \mathcal{G}_{00}(x,x')\, J^0(x')\sqrt{-g(x')}\, d^{n+2}x d^{n+2}x'$$

$$J^0\sqrt{-g} = e\,\delta(x^a - x'^a)$$

$$E_{self}^{em} \to -\frac{1}{2}e^2\, \mathcal{G}_{reg\ 00}(x,x)$$

Hadamard expansion for the Green function

$$G(t,x;t',x') = \int_0^\infty ds\, K(s\,|\,t,x;t',x')$$

$$\left[-\frac{\partial}{\partial s} + U^2\partial_t^2 + U^{-2/n}\Delta\right] K(s\,|\,t,x;t',x') = -U^{-2/n}\delta(t-t')\delta^{n+2}(x-x')\delta(s)$$

$$\mathcal{G}(x,x') = \int_0^\infty ds\, \mathcal{K}(s\,|\,x,x')$$

$$\mathcal{K}(s\,|\,x,x') = \int dt\; K(s\,|\,t,x;t',x')$$

$$\left[-\frac{\partial}{\partial s} + U^{-2/n}\Delta\right] \mathcal{K}(s\,|\,x,x') = -U^{-2/n}\delta^{n+2}(x-x')\delta(s)$$

The Schwinger–DeWitt expansion of the static heat kernel

$$K(s\,|\,x,x') = \frac{D(x,x')^{1/2}}{(4\pi s)^{(n+2)/2}}\ \mathrm{e}^{-\frac{\sigma(x,x')}{2s}} \sum_{k=0}^{\infty} a_k(x,x')\, s^k$$

For the operator

$$\hat{F} = -U^{-2/n}\Delta = g^{ab}\left(\nabla_a - B_a\right)\left(\nabla_b - B_b\right) + V$$

$$g_{ab} = U^{2/n}\delta_{ab} \qquad B_a = \frac{1}{2}\frac{U_a}{U} \qquad V = -B^a B_a + B^a{}_{;a}$$

$$\mathcal{G}_{div}(x,x') = \frac{D(x,x')^{1/2}}{2(2\pi)^{(n+2)/2}} \, \frac{1}{\sigma^{n/2}(x,x')} \sum_{k=0}^{[(n-1)/2]} 2^{-k} \Gamma\left(\frac{n}{2}-k\right) \sigma^{k}(x,x') \, a_k(x,x')$$

If *n=D-3* is an odd integer
plus an additional term $\sim \ln(\sigma)\, a_{n/2}$ **for even *n***

For $\sigma(x,x')$, $a_k(x,x')$ **see, e.g., Eric Poisson Living Rev.Rel. 7 (2004) 6**

$$\mathcal{G}_{reg}(x,x') = \mathcal{G}(x,x') - \mathcal{G}_{div}(x,x')$$

$$E^{sc}_{self} = -\frac{1}{2} q^2 U^{-2}(x) \, \mathcal{G}_{reg}(x,x)$$

In D=4 dimensions (in Schwarzschild coordinates)

$$\mathcal{G}^{sc}(x,x') = \frac{1}{R(x,x')} \longrightarrow \mathcal{G}_{reg}(x,x) = 0$$

$$\mathcal{G}^{em}{}_{00}(x,x') = -\frac{1}{rr'}\left(\frac{\Pi}{R} + M\right) \longrightarrow \mathcal{G}_{00\ reg}(x,x) = -\frac{M}{r^2}$$

Summary

- The exact solutions for static Green functions in the **higher dimensional Majumdar-Papapetrou** metrics was found. It makes possible to treat IR behavior of fields in this background.

- The unambiguous scheme for extracting UV divergencies in the self-energy of static charges was proposed.

$$E^{em}_{self} = -\frac{1}{2} e^2 \; \mathcal{G}_{reg\ 00}(x, x)$$

$$E^{sc}_{self} = -\frac{1}{2} q^2 \; \mathcal{G}_{reg}(x, x) \, U^{-2}(x)$$

$$\mathcal{G}^{sc}(x,x') = \frac{1}{R(x,x')}$$

$$R^2(x,x') = (r-M)^2 + (r'-M)^2 - 2(r-M)(r'-M)\cos\lambda - (M^2-Q^2)\sin^2\lambda$$

$$\cos\lambda = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$\mathcal{G}^{em}(x,x') = -\frac{1}{rr'}\left(\frac{\Pi}{R}+M\right)$$

$$\Pi(x,x') = (r-M)(r'-M) - (M^2-Q^2)\cos\lambda$$

The problem of description of an influence of gravitational field on charged particles has a long history:
J. J. Thomson (1881), Lorentz (1899,1904), Abraham (1904), Poincaré (1905,1906), Fermi (1921) … … … …

Let us try the Tangherlini (higher-dimensional Schwarcshild) metric

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega_{n+1}^2$$

$$f = 1 - \frac{r_0^n}{r^n}, \qquad n = D - 3. \qquad r_0^n = \frac{8\,\Gamma\left(\frac{n}{2}+1\right)G^{(n+3)}}{(n+1)\pi^{n/2}}\mathrm{M}$$

This attempt unfortunately fails for *n>1*. Higher dimensional Reissner-Nordström black hole is not much better.

Extremal Reissner-Nordström black hole

$$ds^2 = -f\,dt^2 + f^{-1/n}\,dr^2 + r^2\,d\Omega^2_{n+1}$$

$$f = \left(1 - \frac{r_0^n}{r^n}\right)^2$$

$$r^n = x^n + r_0^n$$

$$ds^2 = -f\,dt^2 + f^{-1/n}\left(dx^2 + x^2\,d\Omega^2_{n+1}\right)$$

$$f = \frac{x^{2n}}{(x^n + r_0^n)^2}$$

The Green function

$$\left[-f^{-1}\partial_t^2+f^{1/n}\left(x^{-(n+1)}\partial_x(x^{n+1}\partial_x)+x^{-2}\Delta_{\Omega_{n+1}}\right)\right]G(\mathbf{x},\mathbf{x}')=-\delta(\mathbf{x},\mathbf{x}')$$

Static Green function

$$\mathcal{G}(x,x^i;x',x'^i)=\int dt'\,G(\mathbf{x},\mathbf{x}')$$

$$\left[x^{-(n+1)}\partial_x(x^{n+1}\partial_x)+x^{-2}\Delta_{\Omega_{n+1}}\right]\mathcal{G}(x,x^i;x',x'^i)=-\frac{\delta(x-x')\delta(\vec{x}-\vec{x}')}{x^{n+1}\sqrt{h}}$$

Scalar field

$$\mathcal{G}(x, x^i; x', x'^i) = \frac{\Gamma\left(\frac{n}{2}\right)}{4\pi^{1+\frac{n}{2}}} \cdot \frac{1}{R^n}$$

$$R^2 = x^2 + x'^2 - 2xx' \cos\lambda$$

$$\cos\lambda = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\lambda_n$$

$$\cos\lambda_n = \cos\theta_n \cos\theta_n + \sin\theta_n \sin\theta_n \cos\lambda_{n-1}$$

$$\theta_1 = \phi, \quad \theta_{n+1} = \theta, \quad \lambda_{n+1} = \lambda$$

Or in the Schwarzchild coordinates

$$R^2 = \left(r^n - M\right)^{2/n} + \left(r'^n - M\right)^{2/n} - 2\left(r^n - M\right)^{1/n}\left(r'^n - M\right)^{1/n} \cos\lambda$$

$$\cos\lambda = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\lambda_n, \qquad M = r_0^n$$

Maxwell field

$$\frac{1}{\sqrt{-g}}\partial_{\varepsilon}\left(\sqrt{-g}\,g^{00}g^{\varepsilon\beta}\partial_{\beta}A_{0}\right)=-4\pi J^{0}$$

$$\mathcal{G}_{00}(\mathbf{x},\mathbf{x}')=\frac{\Gamma\left(\frac{n}{2}\right)}{4\pi^{\frac{n}{2}+1}}\frac{1}{r^{n}\,r'^{n}}\left[\frac{(r^{n}-M)(r'^{n}-M)}{R^{n}}+M\right]$$

$$M=r_{0}^{n}$$