Analysis of scalar perturbations in nonlocal cosmological models

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A.S. Koshelev, S.Yu. Vernov, Class. Quant. Grav. 28 (2011) 085019, arXiv:1009.0746,
T. Biswas, A.S. Koshelev, T. Mazumdar, and S.Yu. Vernov, 2012, in progress

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To specify different types of cosmic fluids one uses a relation between the pressure p and the energy density ϱ

$$p = w\varrho, \qquad p = E_k - V, \qquad \varrho = E_k + V$$

where w is the state parameter.

In the spatially flat FLRW metric:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

where $a(t)$ is the scale factor, the Hubble parameter $H \equiv \dot{a}/a$,

$$w(t) = -1 + \frac{2E_k}{\varrho} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}. \tag{1}$$

In this case $w_{DE} < -1$ the NEC is violated and there are problems of instability at classical and quantum levels.

A possible way to evade the instability problem for models with $w_{DE} < -1$ is to yield a phantom model as an effective one, arising from a fundamental theory. Such a possibility does appear in the string field theory framework (I.Ya. Aref'eva, 2004).

A particular form of nonlocal gravity has been considered in the context of a phenomenological approach to the cosmological constant problem. The proposal is based on modified Einstein equations at enormous distances

$$M_P^2\left(1+\mathcal{F}(L^2\Box\,)\right)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=\frac{1}{2}T_{\mu\nu} \qquad (2)$$

The function $\mathcal{F}$ satisfies the following conditions

$$\mathcal{F}(L^2\Box)\to 0,\quad at\quad L^2\Box\gg 1 \qquad\qquad \mathcal{F}(z)\to\mathcal{F}(0)\gg 1,\quad at\quad z\to 0. \qquad (3)$$

Arkani-Hamed N., Dimopoulos S., Dvali G., Gabadadze G., 2002.

The combination $M_\Box^2=M_P^2\left(1+\mathcal{F}(L^2\Box\,)\right)$ is treated as "nonlocal" Planck mass.

The left hand side of (2), with nontrivial form factor $\mathcal{F}(L^2\Box\,)$, does not satisfy the Bianchi identity and cannot be represented as a metric variational derivative of the action. However, admitting addition terms of order R^3 to the action it is possible to get (2) [Barvinsky A.O., *Phys. Lett. B* 572 (2003) 109–116].

The Ostragradski representation

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Modified gravity cosmological models have been proposed with the hope to find resolutions to the important open problems of the standard cosmological model. One possible modification which allows to improve the ultraviolet behavior and even to get a renormalizable theory of quantum gravity is the addition of higher-derivative terms to the Einstein–Hilbert action (***K.S. Stelle***, Phys. Rev. D 16 (1977) 953).

Unfortunately, models with the higher-derivative terms have ghosts.

- **M. Ostrogradski**, *Mémoire sur les équations differentielles relatives aux problèmes des isoperimétres*, Mem. St. Petersbourg VI Series, V. 4 (1850) 385–517
- **A. Pais and G.E. Uhlenbeck**, *On Field Theories with Nonlocalized Action*, Phys. Rev. 79 (1950) 145–165

Models with nonlocal scalar fields

The SFT inspired nonlocal gravitation models are introduced as a sum of the SFT action of the tachyon field ϕ plus the gravity part of the action. One cannot deduce this form of the action from the SFT.

Let us consider the $f(R)$ gravity, which is a straightforward modification of the general relativity, and the following action:

$$S_f = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} + \frac{1}{\alpha' g_o^2}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\alpha'\Box\right)\phi - V(\phi)\right) - \Lambda\right), \quad (4)$$

where g_o is the open string coupling constant, α' is the string length squared.

From the SFT after some approximations we obtained:

$$\mathcal{F}_{SFT}(\alpha'\Box) = (\xi^2\alpha'\Box + 1)e^{-2\alpha'\Box} - c,$$

where c and ξ^2 are constants.

$\mathcal{F}_{SFT}$ has only simple and (for some values of c and ξ^2) double roots.

The function $\mathcal{F}$ is assumed to be analytic at all finite points of the complex plane, in other words, to be an entire function. The function $\mathcal{F}$ can be represented by the convergent series expansion:

$$\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n. \tag{5}$$

The Weierstrass factorization theorem asserts that the function $\mathcal{F}$ can be represented by a product involving its zeroes J_k:

$$\mathcal{F}(J) = J^m e^{Y(J)} \prod_{k=1}^{\infty} \left(1 - \frac{J}{J_k}\right) e^{\frac{J}{J_k} + \frac{J^2}{2J_k^2} + \cdots + \frac{1}{p_k}\left(\frac{J}{J_k}\right)^{p_k}}, \tag{6}$$

where m is an order of the root $J = 0$ (m can be equal to zero), $Y(J)$ is an entire function, natural numbers p_n are chosen such that the series $\sum\limits_{n=1}^{\infty} \left(\frac{J}{J_n}\right)^{p_n+1}$ is an absolutely and uniformly convergent one.

Scalar fields ϕ (associated with the open string tachyon) is dimensionless, while $[\alpha'] =$ length^2 and $[g_o] =$ length.

Let us introduce dimensionless coordinates $\bar{x}_\mu = x_\mu/\sqrt{\alpha'}$,
the dimensionless Newtonian constant $\bar{G}_N = G_N/\alpha'$,
the dimensionless $\bar{g}_o = g_o/\sqrt{\alpha'}$.

The dimensionless cosmological constant $\bar{\Lambda} = \Lambda\alpha'^2$, $\bar{R}$ is the curvature scalar in the coordinates $\bar{x}_\mu$:

$$S_f = \int d^4\bar{x}\sqrt{-g}\left(\frac{\bar{R}}{16\pi\bar{G}_N} + \frac{1}{\bar{g}_o^2}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\bar{\Box}_g\right)\phi - V(\phi)\right) - \bar{\Lambda}\right), \qquad (7)$$

In the following formulae we omit bars, but use only dimensionless coordinates and parameters.

Recall the covariant d'Alembertian acting to a scalar:

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{\mu\nu}\nabla_\mu\partial_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right),$$

where ∇_μ is the covariant derivative.

The action is

$$S_f = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} + \frac{1}{g_o^2}\left(\frac{1}{2}\phi\,\mathcal{F}\left(\Box\right)\phi - V(\phi)\right) - \Lambda\right), \qquad (8)$$

The Einstein equations are as follows:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G_N\left(T_{\mu\nu} - \Lambda g_{\mu\nu}\right), \qquad (9)$$

$$\mathcal{F}(\Box)\phi = \frac{dV}{d\phi}, \qquad (10)$$

where the energy–momentum (stress) tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{g_{\mathbf{o}}^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\varrho\sigma}E_{\varrho\sigma} + W\right)\Big), \qquad (11)$$

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\partial_\mu\Box^l\phi\partial_\nu\Box^{n-1-l}\phi, \qquad W \equiv \frac{1}{2}\sum_{n=2}^{\infty} f_n \sum_{l=1}^{n-1}\Box^l\phi\Box^{n-l}\phi - \frac{f_0}{2}\phi^2 + V.$$

The system of the Einstein equations is a system of non-local nonlinear equations !!!

HOW CAN WE FIND A SOLUTION?

There are two different cases:

- **The potential** $V(\phi) = C_2\phi^2 + C_1\phi + C_0$, **where** C_2, C_1 **and** C_0 **are arbitrary constants. In this case one can construct the equivalent action with local fields and quadratic potentials. Number of local fields is equal to number of roots of** $\mathcal{F}(\Box)$, **with a glance of order of them. It has been proved for an arbitrary analytic function** $\mathcal{F}$ **with simple and double roots.**

  I.Ya. Aref'eva, L.V. Joukovskaya, S.Yu.V., ***J. Phys. A: Math. Theor.*** **41 (2008) 304003**, `arXiv:0711.1364`;

  D.J. Mulryne, N.J. Nunes, ***Phys. Rev.*** **D 78 (2008) 063519**, `arXiv:0805.0449`

  S.Yu.V., ***Class. Quant. Grav.*** **27 (2010) 035006**, `arXiv:0907.0468`

  S.Yu.V., ***Phys. Part. Nucl. Lett.*** **8 (2011) 310–320**, `arXiv:1005.0372`

  A.S. Koshelev, S.Yu.V., Class. Quant. Grav. **28 (2011) 085019**, `arXiv:1009.0746`

- **The potential $V(\phi) \neq C_2\phi^2 + C_1\phi + C_0$. In this case situation is more difficult and exact solutions is possible to find only adding some scalar field, for example, a k-essence field.**

  Numerical Solution:

  L. Joukovskaya, *JHEP* 0902 (2009) 045, `arXiv:0807.2065`

  Approximate solutions for field equation:

  G. Calcagni and G. Nardelli, *Int. J. Mod. Phys.* D 19 (2010) 329–338, `arXiv:0904.4245`

  Exact solutions for field equation:

  S.Yu.V., *Theor. Math. Phys.* 166 (2011) 392–402, `arXiv:1005.5007`

An algorithm of localization in the case of an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$.

$$V_{eff} = \left(C_2 - \frac{f_0}{2}\right)\phi^2 + C_1\phi + C_0 + \Lambda. \qquad (12)$$

We can change values of f_0 and Λ such that the potential takes the form $V(\phi) = C_1\phi$.

In other words, we put $C_2 = 0$ and $C_0 = 0$.

There exist 3 cases:

- $C_1 = 0$
- $C_1 \neq 0$ **and** $f_0 \neq 0$
- $C_1 \neq 0$ **and** $f_0 = 0$

I will speak about the case $C_1 = 0$ and assume that all roots of $\mathcal{F}(J)$ are simple.

The general case has been considered in
S.Yu.V., Phys. Part. Nucl. Lett. 8 (2011) 310–320, `arXiv:1005.0372`.

Let us consider the case $C_1 = 0$ and the equation

$$\mathcal{F}(\Box)\phi = 0. \tag{13}$$

We seek a particular solution of (13) in the following form

$$\phi_B = \sum_{i=1}^{N_1} \phi_i. \tag{14}$$

$$(\Box - J_i)\phi_i = 0, \tag{15}$$

J_i **are simple roots of the characteristic equation** $\mathcal{F}(J) = 0$**.**

Energy–momentum tensor for special solutions

If we have *one simple root* ϕ_1 such that $\Box\phi_1 = J_1\phi_1$, then

$$E_{\mu\nu}(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n-1}\partial_\mu\phi_1\partial_\nu\phi_1 = \frac{\mathcal{F}'(J_1)}{2}\partial_\mu\phi_1\partial_\nu\phi_1.$$

$$W(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n}\phi_1^2 = \frac{J_1}{2}\sum_{n=1}^{\infty} f_n n J_1^{n-1}\phi_1^2 = \frac{J_1\mathcal{F}'(J_1)}{2}\phi_1^2.$$

In the case of *two simple roots* ϕ_1 and ϕ_2 we have

$$E_{\mu\nu}(\phi_1+\phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2), \qquad W(\phi_1+\phi_2) = W(\phi_1) + W(\phi_2). \tag{16}$$

In the case of N *simple roots*,

$$T_{\mu\nu} = \sum_{k=1}^{N} \mathcal{F}'(J_k)\left(\partial_\mu\phi_k\partial_\nu\phi_k - \frac{1}{2}g_{\mu\nu}\left(g^{\varrho\sigma}\partial_\varrho\phi_k\partial_\sigma\phi_k + J_k\phi_k^2\right)\right). \tag{17}$$

If $\mathcal{F}(J)$ has simple real roots, then positive and negative values of $\mathcal{F}'(J_i)$ alternate, so we obtain phantom fields.

Considering the following local action

$$S_{loc} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} - \Lambda\right) + \sum_{i=1}^{N_1} S_i, \qquad (18)$$

where

$$S_i = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\frac{\mathcal{F}'(J_i)}{2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + J_i\phi_i^2\right),$$

we can see that solutions of the Einstein equations and equations in ϕ_i, obtained from this action, solves the initial nonlocal equations (9).

Special solutions to nonlocal equations can be found as solutions to system of local (differential) equations.

If $\mathcal{F}(J)$ has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models and the initial nonlocal action (7) generates infinity number of local actions (18).

Scalar Perturbations

Scalar metric perturbations are given by four arbitrary scalar functions $\phi(\eta, x^a)$, $\beta(\eta, x^a)$, $\psi(\eta, x^a)$, $\gamma(\eta, x^a)$ in the following way

$$ds^2 = a(\eta)^2 \left(-(1+2\phi)d\eta^2 - 2\partial_i\beta d\eta dx^i + ((1+2\psi)\delta_{ij} + 2\partial_i\partial_j\gamma)dx^i dx^j\right),$$

where η is the conformal time related to the cosmic one as $a(\eta)d\eta = dt$.

Changing the coordinate system one can both produce fictitious perturbations and remove real ones.

There exist two independent gauge-invariant variables (the Bardeen potentials), which fully determine the scalar perturbations of the metric tensor:

$$\Phi = \phi - \dot{\chi}, \qquad \Psi = \psi - H\chi, \quad \textbf{where} \quad \chi = a\beta + a^2\dot{\gamma}.$$

The gauge invariant variables Φ and Ψ have a very simple physical interpretation: they are amplitudes of the metric perturbations in the longitudinal (conformal-Newtonian) gauge:

$$ds^2 = a(\eta)^2 \left(-(1+2\Phi)d\eta^2 + \delta_{ij}(1+2\Psi)dx^i dx^j\right).$$

The energy–momentum tensor of nonlocal scalar field

The energy–momentum tensor of the nonlocal scalar field is a perfect fluid

$$T^{\mu}_{\nu} = \mathrm{diag}(-\varrho, p, p, p),$$

where

$$\begin{aligned} \varrho \; &= \; \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\Big(\partial_t \Box^l T \partial_t \Box^{n-1-l} T + \Box^l T \Box^{n-l} T\Big) - \\ &\quad - \frac{1}{2} T \mathcal{F}_0(\Box) T + V_{int}(T) + g_o^2 \Lambda_0, \\ p \; &= \; \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\partial_t \Box^l T \partial_t \Box^{n-1-l} T - \Box^l T \Box^{n-l} T\right) + \\ &\quad + \frac{1}{2} T \mathcal{F}_0(\Box) T - V_{int}(T) - g_o^2 \Lambda_0. \end{aligned} \qquad (19)$$

To the perturbed order one has

$$\begin{aligned}
\delta\varrho &= \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Big(\partial_t\delta(\Box^l T)\partial_t\Box^{n-1-l}T + \partial_t\Box^l T\partial_t\delta(\Box^{n-1-l}T) - \\
&\quad - 2\Phi\partial_t\Box^l T\partial_t\Box^{n-1-l}T + \\
&\quad + \delta(\Box^l T)\Box^{n-l}T + \Box^l T\delta(\Box^{n-l}T)\Big) - \frac{1}{2g_o^2}(TV''_{int} - V'_{int})\delta T, \\
\delta p &= \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Big(\partial_t\delta(\Box^l T)\partial_t\Box^{n-1-l}T + \partial_t\Box^l T\partial_t\delta(\Box^{n-1-l}T) - \\
&\quad - 2\Phi\partial_t\Box^l T\partial_t\Box^{n-1-l}T - \\
&\quad - \delta(\Box^l T)\Box^{n-l}T - \Box^l T\delta(\Box^{n-l}T)\Big) + \frac{1}{2g_o^2}(TV''_{int} - V'_{int})\delta T, \\
v^s &= \frac{k}{a(\varrho+p)}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_t\Box^l T\delta\left(\Box^{n-1-l}T\right), \\
\pi^s &= 0.
\end{aligned}$$

Using the Einstein equations one gets that $\pi^s = 0$ is equivalent to $\Phi = \Psi$.

The Bardeen potential Ψ is proportional to the gauge invariant total energy perturbation

$$\varepsilon \equiv \frac{\delta\varrho}{\varrho} + 3(1+\omega)Hv^s\frac{a}{k} = -\frac{k^2}{4\pi G\varrho a^2}\Psi. \tag{20}$$

The function ε is a solution of

$$\begin{aligned} &\ddot{\varepsilon} + H\left(2 + 3c_s^2 - 6w\right)\dot{\varepsilon} + \\ &+\left(\dot{H}(1-3w) - 15H^2w + 9H^2c_s^2 + \frac{k^2}{a^2}\right)\varepsilon + \frac{k^2}{a^2\varrho}\Delta = 0. \end{aligned} \tag{21}$$

Here $w = p/\varrho$, $c_s^2 = \dot{p}/\dot{\varrho}$ is the speed of sound, $k = \sqrt{k_a k^a}$.

$$\begin{aligned} \Delta = \delta p - \delta\varrho + (1-c_s^2)\frac{a}{k}\dot{\varrho}v^s = \frac{(1-c_s^2)\dot{\varrho}}{\varrho+p}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\partial_t\Box^l T\delta(\Box^{n-1-l}T) - \\ -\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\delta(\Box^l T)\Box^{n-l}T + \Box^l T\delta(\Box^{n-l}T)\right) + \frac{1}{g_o^2}(TV''_{int} - V'_{int})\delta T. \end{aligned}$$

The latter quantity is identically zero for a local scalar field, i.e. in the case $\mathcal{F}(\Box) = f_1\Box + f_0$. Therefore, $\Delta \neq 0$ is the attribute of the nonlocality here.

For the linearized model (7) we can consider the background solution as given by (14) to obtain Δ in the more convenient form.

To do this the following relation is useful

$$\delta(\Box^n\phi) = \Box^n\delta\phi + \sum_{m=0}^{n-1}\Box^m(\delta\Box)\Box^{n-1-m}\phi. \tag{22}$$

Using

$$\sum_{m=0}^{n-1} x^m = \frac{x^n - 1}{x - 1},$$

one has

$$\delta(\Box^n\phi) = \Box^n\delta\phi + \sum_i \frac{\Box^n - J_i^n}{\Box - J_i}(\delta\Box)\phi_i. \tag{23}$$

Perturbing the equation of motion for ϕ, one has

$$\delta(\mathcal{F}(\Box)\phi) = \mathcal{F}\sum_i \left(\frac{1}{\Box - J_i}(\delta\Box)\phi_i + \delta\phi_i\right) = 0$$

where we have put $\delta\phi = \sum\limits_i \delta\phi_i$.

It follows from (17) that if for some J_k we have $\phi_k = 0$ as a background solution, then $\delta\phi_k$, contributes only to the second order in the energy–momentum tensor perturbations. We consider perturbations only to the first order, and, therefore, for all $\phi_k = 0$ we can put $\delta\phi_k = 0$ without loss of generality.

$$\Delta = -\frac{2}{\varrho + p}\sum_{m,l}\mathcal{F}'(J_m)\mathcal{F}'(J_l)J_m\phi_m\dot{\phi}_m\dot{\phi}_l^2\zeta_{ml}.$$

where

$$\zeta_{ij} = \frac{\delta\phi_i}{\dot{\phi}_i} - \frac{\delta\phi_j}{\dot{\phi}_j}$$

The functions ζ_{ij} satisfy the following set of equations

$$\begin{aligned}
&\ddot{\zeta}_{ij} + \left(3H + \frac{\ddot{\phi}_i}{\dot{\phi}_i} + \frac{\ddot{\phi}_j}{\dot{\phi}_j}\right)\dot{\zeta}_{ij} + \left(-3\dot{H} + \frac{k^2}{a^2}\right)\zeta_{ij} = \\
&= \left(\frac{J_i\phi_i}{\dot{\phi}_i} - \frac{J_j\phi_j}{\dot{\phi}_j}\right)\left(\sum_m \frac{\mathcal{F}'(J_m)\dot{\phi}_m^2}{\varrho + p}\left(\dot{\zeta}_{im} + \dot{\zeta}_{jm}\right) + \frac{2}{1+w}\varepsilon\right).
\end{aligned} \tag{24}$$

Equation (21) with the above derived Δ and equations (24) describe the perturbations in the case of linearized model.

Note that only $N-1$ functions ζ_{1j} are independent.

So, in the case of quadratic potential analysis of the first order perturbation is equivalent to analysis of perturbations in the model with local scalar fields with quadratic potentials.

NONLOCAL GRAVITY

A nonlocal modification that assumes the existence of a new dimensional parameter M_* can be of the form

$$S=\int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R+\frac{1}{2}R\mathcal{F}\left(\frac{\Box}{M_*^2}\right)R-\Lambda\right) \qquad (25)$$

where M_* is the mass scale at which the higher derivative terms in the action become important, $8\pi G_N=1/M_P^2$**.**

An analytic function $\mathcal{F}(\Box/M_*^2)=\sum\limits_{n\geqslant 0} f_n\Box^n$.

Biswas T., Mazumdar A., and Siegel W., 2006, *JCAP* 0603 009 (hep-th/0508194)

Biswas T., Koivisto T., and Mazumdar T., 2010, *JCAP* 1011 008 (arXiv:1005.0590)

Koshelev A.S., Vernov S.Yu., arXiv:1202.1289,

Biswas T., Koshelev A.S., Mazumdar T., and S.Yu.V., 2012, in progress.

By virtue of the field redefinition one can transform the non-local gravity action (25) as follows:

$$S=\int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}(1+\Phi)R+\frac{1}{2}\phi\mathcal{F}\left(\frac{\Box}{M_*^2}\right)\phi-\frac{M_P^2}{2}\Phi\phi-\Lambda\right) \qquad (26)$$

with two new scalar fields Φ and ϕ.

Variation w.r.t. Φ gives $\phi=R$ and, therefore, the connection (26) with action (25) is obvious.

From action (26) one gets the following equations of motion:

$$M_P^2(1+\Phi)\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\frac{1}{2}\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\Big(\partial_\mu\Box^l\phi\partial_\nu\Box^{n-1-l}\phi+$$

$$+\partial_\nu\Box^l\phi\partial_\mu\Box^{n-1-l}\phi-g_{\mu\nu}\left(g^{\varrho\sigma}\partial_\varrho\Box^l\phi\partial_\sigma\Box^{n-1-l}\phi+\Box^l\phi\Box^{n-l}\phi\right)\Big)+$$

$$+\frac{1}{2}g_{\mu\nu}\left(\phi\mathcal{F}(\Box)\phi-M_P^2\Phi\phi\right)+M_P^2(D_\mu\partial_\nu\Phi-g_{\mu\nu}\Box\Phi)-\Lambda g_{\mu\nu}\,,$$

$$\mathcal{F}(\Box)\phi=\frac{M_P^2}{2}\Phi\,,$$

$$\phi=R\,.$$

Equations in Non-local Gravity Models

Variation of action (25) yields the following equations

$$
\begin{aligned}
&[M_P^2 + 2\mathcal{F}(\Box)R]G^\mu_\nu = \\
&= \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\Big[g^{\mu\varrho}\partial_\varrho\Box^l R\partial_\nu\Box^{n-l-1}R + g^{\mu\varrho}\partial_\nu\Box^l R\partial_\varrho\Box^{n-l-1}R - \\
&- \delta^\mu_\nu\left(g^{\varrho\sigma}\partial_\varrho\Box^l R\partial_\sigma\Box^{n-l-1}R + \Box^l R\Box^{n-l}R\right)\Big] - \frac{1}{2}R\mathcal{F}(\Box)R\delta^\mu_\nu + \\
&+ 2(g^{\mu\varrho}\nabla_\varrho\partial_\nu - \delta^\mu_\nu\Box)\mathcal{F}(\Box)R - \Lambda\delta^\mu_\nu + {T_{\mathrm{M}}}^\mu_\nu,
\end{aligned}
\tag{27}
$$

where ${T_{\mathrm{M}}}^\mu_\nu$ **is the energy–momentum tensor of matter and**

$$
G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R \ , \tag{28}
$$

is the Einstein tensor.

The equations can be written in a compact form:

$$
\begin{aligned}
&[M_P^2 + 2\mathcal{F}(\Box)R]G^\mu_\nu = {T_{\mathrm{M}}}^\mu_\nu - \Lambda\delta^\mu_\nu + \\
&+\mathcal{K}^\mu_\nu - \frac{}{2}\delta^\mu_\nu\left(\mathcal{K}^\sigma_\sigma + \mathcal{K}_1\right) - \frac{}{2}R\mathcal{F}(\Box)R\delta^\mu_\nu + 2(g^{\mu\varrho}\nabla_\varrho\partial_\nu - \delta^\mu_\nu\Box)\mathcal{F}(\Box)R\ ,
\end{aligned}
\tag{29}
$$

where we have introduced two additional quantities:

$$
\mathcal{K}^\mu_\nu = g^{\mu\varrho}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\varrho\Box^l R\partial_\nu\Box^{n-l-1}R\,, \qquad \mathcal{K}_1 = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Box^l R\Box^{n-l}R\,.
\tag{30}
$$

The trace equation is

$$
M_P^2 R - \mathcal{K}^\mu_\mu - 2\mathcal{K}_1 - 6\Box\mathcal{F}(\Box)R = 4\Lambda - {T_{\mathrm{M}}}^\mu_\mu\,,
\tag{31}
$$

General Ansatz for finding Exact Solutions

The following ansatz

$$\Box R = r_1 R + r_2, \qquad r_1 \neq 0 \ , \tag{32}$$

is useful in finding exact solutions. From (32) one gets

$$\mathcal{F}(\Box) R = \mathcal{F}_1 R + \frac{r_2}{r_1}(\mathcal{F}_1 - f_0) \quad \textbf{where} \quad \mathcal{F}_1 \equiv \mathcal{F}(r_1).$$

If the scalar curvature R satisfies (32), then the trace equation is

$$A_1 R - \mathcal{F}'(r_1)\left(2 r_1 R^2 + \partial_\mu R \partial^\mu R\right) + A_2 = T_{\mathrm{M}}{}^{\mu}_{\mu} \,, \tag{33}$$

where

$$A_1 = M_P^2 - \left(4\mathcal{F}'(r_1) r_2 - 2\frac{r_2}{r_1}(\mathcal{F}_1 - f_0) + 6\mathcal{F}_1 r_1\right),$$

$$A_2 = \ -4\Lambda - \frac{r_2}{r_1}\left(2\mathcal{F}'(r_1) r_2 - 2\frac{r_2}{r_1}(\mathcal{F}_1 - f_0) + 6\mathcal{F}_1 r_1\right).$$

We consider only a traceless perfect fluid (radiation):$T_{\mathrm{M}}{}^{\mu}_{\mu} = 0$.

The simplest way to get a solution to (33) with ${T_{\mathrm{M}}}^{\mu}_{\mu} = 0$ **is to put** $A_1 = A_2 = 0$ **and impose** $\mathcal{F}^{(1)}(r_1) = 0$**. This implies**

$$r_2 = -\frac{r_1[M_P^2 - 6\mathcal{F}_1 r_1]}{2[\mathcal{F}_1 - f_0]}, \qquad \Lambda = -\frac{r_2 M_P^2}{4r_1}.$$

The Einstein equations are simplified to

$$2\mathcal{F}_1(R + 3r_1)G^{\mu}_{\nu} = {T_{\mathrm{M}}}^{\mu}_{\nu} + 2\mathcal{F}_1\left[g^{\mu\varrho}\nabla_{\varrho}\partial_{\nu}R - \frac{1}{4}\delta^{\mu}_{\nu}\left(R^2 + 4r_1 R + r_2\right)\right] \qquad (34)$$

Using the ansatz, one can find exact non-singular bouncing solutions in the FLRW metric:

$$a(t) = a_0 \cosh(\lambda t), \qquad (35)$$

where a_0 **is an arbitrary constant and** $\lambda = \sqrt{\Lambda}/3M_P$**. To satisfy all Einstein equations one should add a radiation to the model.**

Another bouncing solution is

$$a(t) = a_0 e^{\frac{\lambda}{2}t^2}, \qquad (36)$$

$$H(t) = \lambda t, \quad R = 12\lambda^2 t^2 + 6\lambda, \quad \Lambda = \lambda M_P^2/2, \quad r_1 = -6\lambda, \quad r_2 = 12\lambda^2,$$

Note that function (36) is a solution to all Einstein equations.

Perturbations

To compute perturbation equations we use the fact that the background configuration obeys the ansatz and the corresponding conditions on parameters.

Let us introduce the following notations to separate the background and perturbations:

$$g_{\mu\varrho} = g_{B\mu\varrho} + h_{\mu\varrho}\,, \qquad R = R_B + \delta R, \qquad \Box = \Box_B + \delta\Box,$$

Also, let us denote the variation of the ansatz (32) as follows:

$$\zeta \equiv \delta\Box R_B + (\Box_B - r_1)\delta R. \tag{37}$$

While computing the variation of the Einstein equations, what we encounter most often is the variation

$$\delta(\Box^n R) = \sum_{l=0}^{n-1} \Box_B^l \delta\Box \Box_B^{n-l-1} R_B + \Box_B^n \delta R \,,$$

which thanks to ansatz (32) sums up to

$$\delta(\Box^n R) = \frac{\Box_B^n - r_1^n}{\Box_B - r_1} \delta\Box R_B + \Box_B^n \delta R = \frac{\Box_B^n - r_1^n}{\Box_B - r_1} \zeta + r_1^n \delta R \,, \tag{38}$$

and this is a key simplification.

We obtain

$$\delta(\mathcal{F}(\Box) R) = \frac{\mathcal{F}(\Box_B) - \mathcal{F}_1}{\Box_B - r_1} \zeta + \mathcal{F}_1 \delta R \equiv (\Box_B - r_1)\Xi + \mathcal{F}_1 \delta R \,. \tag{39}$$

To get the Einstein equations for perturbations we also use

$$\begin{aligned} \delta\mathcal{K}^\mu_\nu &= {g_B}^{\mu\varrho} \left(\partial_\varrho \Xi \partial_\nu R_B + \partial_\varrho R_B \partial_\nu \Xi\right) , \\ \delta\mathcal{K}_1 &= 2\Xi \left(r_1 R_B + r_2\right) + R_B(\Box_B - r_1)\Xi - \frac{r_2}{r_1} \delta R(\mathcal{F}_1 - f_0). \end{aligned} \tag{40}$$

The Trace Equation for Perturbations

Assuming $\delta T_{\mathrm{M}}{}^{\mu}_{\mu} = 0$**,**

we get trace equation for perturbations as follows:

$$M_P^2 \delta R - \delta \mathcal{K}^{\mu}_{\mu} - 2\delta \mathcal{K}_1 - 6\delta(\Box \mathcal{F}(\Box) R) = 0. \tag{41}$$

Using condition $A_1 = 0$ **to eliminate** M_P^2**:**

$$M_P^2 + 2\mathcal{F}(\Box_B) R_B = 2\mathcal{F}_1 (R_B + 3r_1),$$

we obtain that equation (41) is a linear equation in ζ**:**

$$g_B^{\mu\nu} \partial_\nu R_B \partial_\mu \Xi + r_1 R_B \Xi + 2r_2 \Xi + R_B \Box_B \Xi + 3[\Box_B(\Box_B - r_1)\Xi + \mathcal{F}_1 \zeta] = 0, \tag{42}$$

where

$$\Xi = \frac{\mathcal{F}(\Box) - \mathcal{F}_1}{(\Box - r_1)^2} \zeta.$$

The explicit calculations show that ζ **is gauge-invariant function, therefore, it is not possible to choose such gauge that** $\zeta = 0$**.**

The Bardeen potentials and the matter perturbations

For scalar perturbations, in terms of the Bardeen potentials, the function ζ can be written as:

$$\zeta = \frac{R'_B}{a^2}(\Phi' - 3\Psi') - 2(r_1 R_B + r_2)\Phi + (\Box_B - r_1)\,\delta R_{\mathbf{GI}}. \tag{43}$$

where $\delta R_{\mathbf{GI}}$ is the gauge invariant part of the curvature variation:

$$\delta R_{\mathbf{GI}} = \frac{2}{a^2}\left[k^2(\Phi + 2\Psi) - 3\frac{a'}{a}\Phi' - 6\frac{a''}{a}\Phi + 3\Psi'' + 9\frac{a'}{a}\Psi'\right].$$

From the (i,j) component of system of the perturbed Einstein equations, with $i \neq j$, we get the following equation

$$(\Box_B - r_1)\Xi + \mathcal{F}_1[\delta R_{\mathbf{GI}} + (R_B + 3r_1)(\Phi + \Psi)] = 0. \tag{44}$$

So, there are two nonlocal equations for the two Bardeen potentials, Φ and Ψ.

The most general form of the firth-order scalar perturbations of the energy–momentum tensor in the fluid notations can be parameterized by four scalar function:

$$T^0_0 = -(\varrho + \delta\varrho), \quad T^0_i = -\frac{1}{k}(\varrho + p)\partial_i v^s, \quad T^i_j = (p + \delta p)\delta^i_j + \left(\frac{\partial^i \partial_j}{k^2} + \frac{\delta^i_j}{3}\right)\pi^s,$$

where ϱ is the energy density, p the pressure, v^s the velocity or the flux related variable and π^s the anisotropic stress.

We consider only radiation for which

$$\pi^s = 0, \qquad \delta p = \frac{1}{3}\delta\varrho \ .$$

The function v^s and $\delta\varrho$ can be obtained from the $(0,0)$ and $(0,j)$ Einstein equations as follows:

$$\delta\varrho = \left[\frac{R_B'}{a^2}\Xi' + (r_1 R_B + r_2)\Xi + R_B(\Box - r_1)\Xi + R_B\mathcal{F}_1\delta R_{\mathbf{GI}} + \right.$$
$$+2\mathcal{F}_1(\zeta + r_1\delta R_{\mathbf{GI}}) - 4\mathcal{F}_1(R_B + 3r_1)\left(3\frac{a'(\Psi' - \mathcal{H}\Phi)}{a^3} + \frac{k^2}{a^2}\Psi\right) -$$
$$\left. - 6\frac{a'^2}{a^4}\Upsilon + \frac{2}{a^2}\left(\Upsilon'' - \mathcal{H}\Upsilon'\right) + 2\Box_B(\Box_B - r_1)\Xi\right]$$

and

$$v^s = \frac{3k}{2\varrho a^2}\left[2\mathcal{F}_1(R_B + 3r_1)\left(\Psi' - \mathcal{H}\Phi\right) + \; + \frac{R'}{2}\Xi + \Upsilon' - \mathcal{H}\Upsilon\right],$$

where

$$\Upsilon = (\Box_B - r_1)\Xi + \mathcal{F}_1\delta R_{\mathbf{GI}}.$$

A particular solution of the perturbation equations: $\zeta = 0$

The perturbation equations are nonlocal equations, it is difficult to get the general solution.

However, they has a particular solution, corresponding to $\zeta = 0$.

In this case, we obtain a system of two second order differential equations for the Bardeen potentials:

$$\Phi + \Psi = -\frac{\delta R_{\mathbf{GI}}}{R_B + 3r_1} \tag{45}$$

$$\frac{R_B'}{a^2}(\Phi' - 3\Psi') - 2(r_1 R_B + r_2)\Phi = (\Box - r_1)\,(R_B + 3r_1)(\Phi + \Psi). \tag{46}$$

Remarkably, that the same system can be obtained in $f(R)$ gravity model.

Indeed, in the case of $\zeta = 0$, ansatz (32) is valid not only the background solution $\Box R_B$, but also for perturbations.

It means that one can use equations (34):

$$2\mathcal{F}_1(R+3r_1)G^{\mu}_{\nu} = T_{\mathrm{M}}{}^{\mu}_{\nu} + 2\mathcal{F}_1\left[g^{\mu\varrho}\nabla_{\varrho}\partial_{\nu}R - \frac{1}{4}\delta^{\mu}_{\nu}\left(R^2+4r_1R+r_2\right)\right] \ ,$$

to get the perturbation equations. These equations coincide the equations of $f(R)$ gravity model with the action

$$S_f = \int d^4x\sqrt{-g}\left(\frac{\mathcal{F}_1}{2}\left[R^2+6r_1R+3r_2\right]+\mathcal{L}_{\mathrm{M}}\right), \qquad (47)$$

where $\mathcal{L}_{\mathrm{M}}$ is the Lagrangian of the radiation.

Note that, in distinguish to the f(R) gravity, in the considering model equations (45) and (46) does not describe all possible scalar perturbations.

Conclusion

- **There are a lot of possibilities to include nonlocality in cosmological models. We hope that nonlocal cosmological models can be constructed as an effective action inspired by the string field theory.**
- **In the case of the general relativity model minimally coupling with nonlocal scalar field we construct the equation for the energy density perturbations of the nonlocal scalar field in the presence of the arbitrary potential and formulate the local system of equations for perturbations for the linearized model.**
- **For nonlocally modified model with radiation we construct the perturbation equations and show that the four perturbation equations can be separate on two nonlocal equations for the Bardeen potentials and to trivial equations for the perturbations of the matter energy-stress tensor.**

THANK YOU!