The ultraviolet behaviour $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravities

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based on [arXiv:1202.3692] with
Piotr Tourkine
and [arXiv:1105.6087] with
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Motivations

$\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity arises as the low-energy limit of string

String theory provides a consistent ultraviolet finite theory of quantum gravity. One could wonder if one can remove the string massive modes and address the question of ultraviolet behaviour of *pure supergravity*

In this talk we will discuss

- the role of supersymmetry in perturbative computation
- the role of non-perturbative duality symmetries in string theory

Behavior of supergravity amplitudes

Gravity has a dimensional coupling constant

$$[1/\kappa^2_{(D)}] = \text{mass}^{D-2}$$

An L-loop n-point gravity amplitude in D-dimensions has the dimension

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4-graviton amplitudes factorize an $\mathcal{R}^4$ term and possibly higher derivatives

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Supersymmetry and UV behaviour

- Critical dimension for UV divergence is

$$D \geqslant D_c = 2 + \frac{6 + 2\beta_L^N}{L}$$

- Depending on the various implementations of supersymmetry

$$6 \leqslant 6 + 2\beta_L^N \leqslant 18$$

- With a first possible divergence in $D = 4$ at
  - $L \geqslant 3$: $\beta_L^N = 0$ [Howe, Lindstrom, Stelle '81]
  - $L \geqslant 5$: $\beta_L^8 = 2$ [Howe, Stelle '06; Bossard, Howe, Stelle '09]
  - $L \geqslant 8$: $\beta_L^8 = 5$ [Kallosh '81]
  - $L \geqslant 7$: $\beta_L^8 = 4$ [Vanhove '10; Green, Bjornsson '10]
  - $L \geqslant 9$: $\beta_L^8 = 6$ [Green,Russo,Vanhove '06]
  - $L = \infty$: $\beta_L^8 = L$ [Green,Russo,Vanhove '06]

Non-renormalisation theorems

Supersymmetry implies various non-renormalisation theorems for higher dimension operators.

- Heterotic compactification ($\mathcal{N} = 4$ models)
  - $\mathcal{R}^4$ is a $\frac{1}{2}$-BPS coupling 1-loop exact in perturbation [Bachas, Kiritsis; Bachas, Fabre, Kiritsis, Obers, Vanhove; Tourkine, Vanhove]
- Type II compactifications on a torus ($\mathcal{N} = 8$ models) [Green, Russo, Vanhove; Berkovits]
  - $\mathcal{R}^4$ is $\frac{1}{2}$-BPS : 1-loop exact
  - $\partial^4\mathcal{R}^4$ is $\frac{1}{4}$-BPS : 2-loop exact
  - $\partial^6\mathcal{R}^4$ is $\frac{1}{8}$-BPS : 3-loop exact

These operators are potential UV divergences counter-term to supergravity in various dimensions

How these stringy results allow to conclude about the ultraviolet behaviour of supergravity?

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity

$\mathcal{N} = 8$ non-renormalisation theorems imply that [Green, Russo, Vanhove]

- 1-loop non-renormalisation of $\mathcal{R}^4$: $\beta_L^8 \geqslant 2$ for $L \geqslant 2$
- 2-loop non-renormalisation of $\partial^4\mathcal{R}^4$: $\beta_L^8 \geqslant 3$ for $L \geqslant 3$
- 3-loop non-renormalisation of $\partial^6\mathcal{R}^4$: $\beta_L^8 \geqslant 4$ for $L \geqslant 4$

Same critical UV behaviour for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA at $1 \leqslant L \leqslant 4$ loops

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-4)L-6}\, \partial^{2L}\, \mathcal{R}^4 \qquad 2 \leqslant L \leqslant 4$$

- This has been confirmed using various field theory supersymmetry [Bossard, Stelle, Howe], and direct loop computation [Bern, Carrasco, Dixon, Johansson, Roiban], and continuous E_7 arguments [Elvang, Keirmaier, Freedman et al.]

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity

- Up to an including 4 loops the rule $\beta_L^8 = L$ is satisfied.
- If this rule is true to all order then the theory would have critical UV behaviour and making the $\mathcal{N} = 8$ SUGRA UV finite in $D = 4$.

$$D_c = 4 + \frac{6}{L}$$

The question is therefore to see a deviation from the $\beta_L^8 = L$ rule
At which order $\mathcal{N} = 8$ SUGRA can have a worse UV behaviour of $\mathcal{N} = 4$ SYM?

The ultraviolet behaviour of $\mathcal{N} = 8$ supergravity

In $\mathcal{N} = 8$ the big question is if the $\partial^8 \mathcal{R}^4$ is protected or not

- After 4-loop it is expected a worse UV behaviour than for $\mathcal{N} = 4$ SYM

[Green, Russo, Vanhove], [Vanhove], [Green, Bjornsson]

$$[\mathfrak{M}_{4;L}^{(D)}] \sim \Lambda^{(D-2)L-14}\, \partial^8\, \mathcal{R}^4 \qquad \beta_L^8 = 4 \text{ for } L \geqslant 4$$

- At five-loop order the 4-point amplitude in
  - $\mathcal{N} = 4$ SYM divergences for $5 < 26/5 \leqslant D$
  - $\mathcal{N} = 8$ SUGRA divergences for $24/5 \leqslant D$

- Would imply a *seven-loop* divergence in $D = 4$ with counter-term $\partial^8 \mathcal{R}^4$

The candidate counter-term $\partial^8 \mathcal{R}^4$ has dimension 16 and it is tempting to conclude that it is a D-term given by the volume of superspace

$$\partial^8\, \mathcal{R}^4 \sim \int d^{32}\theta\, E(x, \theta)$$

Duality invariant superspace volume

Classical extended supergravity is invariant under continuous duality group : $E_{7,7}(\mathbb{R})/SU(8)$ for $\mathcal{N}=8$ and $SU(1,1,\mathbb{R})/U(1)$ for $\mathcal{N}=4$

We can define *duality invariant* superspace volume

$$\int d^{4\mathcal{N}}\theta\, E(x,\theta) = \partial^{2(\mathcal{N}-4)}\, \mathcal{R}^4 + \cdots$$

- For $\mathcal{N}=8$ this gives the candidate 7-loop counter-term in $D=4$: $\partial^8\mathcal{R}^4$
- For $\mathcal{N}=4$ this gives the candidate 3-loop counter-term in $D=4$: $\mathcal{R}^4$

One could be tempted to conclude that this settles the question since there are obvious D-terms

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Доверяй, но проверяй!

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- For $\mathcal{N}=4$ this gives the candidate 3-loop counter-term in $D=4$: $\mathcal{R}^4$
- Extending to $\mathcal{N}\geqslant 4$ a flow equation by [Kuzenko et al.] we showed the *vanishing of the duality invariant superspace volume*

  [Bossard, Howe, Stelle, Vanhove]

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta) = 0, \qquad 4\leqslant\mathcal{N}\leqslant 8$$

Harmonic superspace

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta) = 0, \qquad 4 \leqslant \mathcal{N} \leqslant 8$$

This does not imply the absence of divergence because using harmonic superspace one can construct *fully supersymmetric and duality invariant counterterm* [Bossard, Howe, Stelle, Vanhove]

- We defined the $1/\mathcal{N}$ harmonic measure (over $4(\mathcal{N}-1)$ θs) $d\mu_{(\mathcal{N},1,1)}$ for $U(1)\times U(\mathcal{N}-2)\times U(1)\backslash U(\mathcal{N})$

$$\int d^4x d^{4\mathcal{N}}\theta E(x,\theta)\,\Phi(x,\theta) =: \int d^4x\, d\mu_{(\mathcal{N},1,1)}\,(D^1)^2(\bar{D}_{\mathcal{N}})^2\Phi$$

$1/\mathcal{N}$ Harmonic superspace

With this measure of integration we constructed candidate $\mathcal{N} - 1$-loop counter-terms in $D = 4$ [Bossard, Howe, Stelle, Vanhove]

- The $\partial^8\mathcal{R}^4$ term for $\mathcal{N} = 8$ (χ_α^{ijk} are the mass dimension $\frac{1}{2}$ spinor)

$$\int d\mu_{(8,1,1)}\, \bar\chi^{1mn}\chi_{8mn}\bar\chi^{1pq}\chi_{8pq} \sim \int d^4xe\,(\partial^8\,\mathcal{R}^4 + \cdots)$$

- Supersymmetric and $E_{7(7)}$ invariant because this is expressed in terms of the dim $\frac{1}{2}$ superfield χ ($\frac{1}{2}$-torsion component)

- The $\mathcal{R}^4$ term for $\mathcal{N} = 4$

$$\int d\mu_{(4,1,1)}\, \bar\chi^{1mn}\chi_{4mn}\bar\chi^{1pq}\chi_{4pq} \sim \int d^4xe\,(\mathcal{R}^4 + \cdots)$$

- Supersymmetric and $SU(1,1)$ invariant expression

The special case of $\mathcal{N} = 4$ supergravity

$\mathcal{N} = 4$ supergravity is special because of the $U(1)$ R-symmetry anomaly [Marcus].

Therefore the $SU(1,1)$ duality symmetry is broken in perturbation and full superspace integrals of functions of the axion-dilaton $\mathcal{S} \in SU(1,1)/U(1)$ are allowed

$$\int d^{16}\theta\, E(x,\theta)\, G(\mathcal{S},\bar{\mathcal{S}}) = f(S,\bar{S})\,\mathcal{R}^4 + \text{susy completion}$$

Only for $f = 1$ this is a 3-loop UV divergence counter-term in the 4 graviton amplitude.

But $f = 1$ would violate the $\mathcal{R}^4$ 1-loop non renormalisation theorems derived from string theory

The special case of $\mathcal{N} = 4$ supergravity

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Because of the anomaly canceling term $\Re\mathrm{e} \int d^4x\, h(S,\bar S)\, \mathrm{tr}(R - i * R)^2$ it is tempting to conclude that the $\mathcal{R}^4$ will also have a non-trivial dependence on the scalar field S

$$\kappa_{(4)}^4 \int d^4x f(S,\bar S)\, \mathcal{R}^4$$

This would be compatible with the string theory non-renormalisation theorems

The ultraviolet behavior of $\mathcal{N} = 4$ supergravity

- 4 graviton amplitude computations gives that [Tourkine, Vanhove]

$$M_4^{1-loop} \sim \mathcal{R}^4 I_{box}[\ell^4] \quad : \qquad \beta_1^4 = 0$$
$$M_4^{2-loop} \sim \partial^2 \mathcal{R}^4 I_{double-box}[\ell^4] \quad : \qquad \beta_2^4 = 1$$

- 1-loop non-renormalisation of $\mathcal{R}^4$: $\beta_L^4 \geqslant 1$ for $L \geqslant 2$

First UV divergence in 4D: $\qquad L \geqslant 3 + \beta_L^4 \geqslant 4$ loops

$\mathcal{N} = 4$ non-renormalisation theorems for $\mathcal{R}^4$ term $\beta_L^4 = 1$ for $L \geqslant 2$ [Tourkine, Vanhove]

$$[\mathfrak{M}_L^{(D)}] = \text{mass}^{(D-2)L-8}\, \partial^2 \mathcal{R}^4 \qquad \text{for } L \geqslant 2$$

- The same result was obtained from direct field theory computation [Bern, Davies, Dennen, Huang] (see [Kallosh] for alternative arguments)

Outlook

- Using string theory we have put constraints on the possible counter-terms for UV divergences of four gravitons amplitudes in $\mathcal{N}=4$ and $\mathcal{N}=8$ supergravity

- Using harmonic superspace we have constructed *supersymmetric duality invariant* candidate counter-terms for first possible UV divergence in $D=4$

- We showed that the $\mathcal{R}^4$ term satisfies a *non-renormalisation* theorem in $\mathcal{N}=4$

$$\kappa_{(4)}^4 \int d^4x f(S)\, \mathcal{R}^4$$

- Where $f(S) = tree + 1 - loop$ and no constant piece

- Could it be as well that that the partial superspace $\partial^8\mathcal{R}^4$ in $\mathcal{N}=8$ has an enhanced protection delaying the UV divergences of $\mathcal{N}=8$ supergravity beyond 7-loops ? May be till 9 loops ? [Green, Russo, Vanhove]