Higher-Spin Interactions: three-point functions and beyond

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Based on:

hep-th/1107.5843: M.T.

hep-th/1110.5918: E.Joung and M.T.

hep-th/1203.6578: E.Joung, L. Lopez, M.T.

and also on:

Master Thesis (2009) [hep-th/1005.3061] M.T. (Advisor: A.Sagnotti)

hep-th/1006.5242: A.Sagnotti and M.T.

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Higher Spins and String Theory

✓String Theory is a "scheme" based on a vibrating relativistic string.

✓Although very natural it raises several questions: Background (in)dependence(?) ....

Key ingredient for consistency: Infinite tower of massive Higher-Spin (HS) excitations

String Theory is a consistent HS Theory!

Many efforts to understand the systematics of HS (mostly massless):

Metric-like:

Singh and Hagen 1974, Fronsdal 1978
de Wit and Freedman 1980, Bengtsson 1986
Metsaev 1993-, Buchbinder et al. 1998-
Francia and Sagnotti 2002-
Fotopoulos and Tsulaia 2007-
Boulanger at al. 2008-, Zinoviev 2009-, ...

Frame-like:

Fradkin and Vasiliev 1987, Vasiliev 1989-
Sezgin and Sundell 1998-, Didenko 2003-,
Alkalaev and Gragoriev 2005-, Iazeolla and Sundell 2005-,
Skvortsov 2006-, Zinoviev 2006-, Boulanger 2009-, ...

Vasiliev's System!

Plan

- Noether procedure
- Ingredients
  - Strategy
  - Generating functions
  - Ambient space formalism
- Cubic interactions in any constant curvature background
  - Massless interactions
  - Massive and partially-massless interactions
- Beyond cubic interactions
- Outlook

Noether Procedure

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} + \dots$$

$$\delta\varphi = \delta^{(0)}\varphi + \delta^{(1)}\varphi + \delta^{(2)}\varphi + \dots$$

Free part:

$$\mathcal{L}^{(2)} \sim \varphi \Box \varphi + \dots$$

Fronsdal, ...

Fully non-linear deformation

Usual linearized gauge transformations: $\delta^{(0)}\varphi = D\,\varepsilon$

Finding a solution order by order!

$$\delta\mathcal{L}^{(2)} \sim \Box\varphi + \dots \approx 0 \qquad \delta^{(1)}\mathcal{L}^{(2)} + \delta^{(0)}\mathcal{L}^{(3)} = 0$$

$$\delta^{(2)}\mathcal{L}^{(2)} + \delta^{(1)}\mathcal{L}^{(3)} + \delta^{(0)}\mathcal{L}^{(4)} = 0$$

$$\delta^{(3)}\mathcal{L}^{(2)} + \delta^{(2)}\mathcal{L}^{(3)} + \delta^{(1)}\mathcal{L}^{(4)} + \delta^{(0)}\mathcal{L}^{(5)} = 0$$

......

Non homogeneous pieces start appearing from the fourth order on

Increasingly complicated as soon as higher orders are considered...

...what is the logic behind that?

General Strategy

At the free level a general class of HS actions looks like (massless case):

$$S^{(2)} = \frac{1}{2}\int d^dx \left[\underbrace{\varphi_{\mu_1\ldots\mu_s}\Box\, \varphi^{\mu_1\ldots\mu_s}}_{\text{Transverse Traceless (TT) part}} + \ldots \right]$$

Gauge invariant completion of the TT part!

$$\delta\varphi_{\mu_1\ldots\mu_s}(x) = \partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)}(x)$$

Gauge invariance "fixes" the completion while the TT part is well defined as the equivalence class modulo traces and divergences of the fields!

$$\boxed{\delta_\epsilon \mathcal{L}_{TT} \approx 0}_{TT}$$

Strategy: solve first the simpler problem of finding the TT part of the Lagrangian and afterwards complete it!

arXiv:1003.2877: Manvelyan, Mkrtchyan, Ruehl; hep-th/1006.5242: A.Sagnotti and M.T.

In general many information can be extracted already at this simpler (although incomplete) level!

Generating Functions

Encode all totally symmetric polarization tensors into a single function

$$\varphi_i(x, u_i) = \sum_n \frac{1}{n!} \varphi_{i\,\mu_1 \cdots \mu_n} u_i^{\mu_1} \ldots u_i^{\mu_n} = \varphi_i + u_i^{\mu} \varphi_{i\,\mu} + \frac{1}{2} u_i^{\mu_1} u_i^{\mu_2} \varphi_{\mu_1 \mu_2} + \ldots$$

In ST amounts to the simplification:

$$\alpha_{-1}^{\mu} \to u_{\mu}$$

Commuting variables

Generating Function

$$\mathcal{L}^{(n)} = \mathcal{K}_n(u_i', \partial_{x_i}) \star_{1 \ldots n} [\varphi_1(x_1, u_1) \ldots \varphi_n(x_n, u_n)] \Big|_{x_i = x}$$

$$\phi(u_1) \star_1 \psi(u_1') = \exp\left(\frac{\partial}{\partial u_1} \cdot \frac{\partial}{\partial u_1'}\right) \phi(u_1) \psi(u_1') \Big|_{u=0} = \phi \psi + \phi^{\mu} \psi_{\mu} + \frac{1}{2} \phi^{\mu_1 \mu_2} \psi_{\mu_1 \mu_2} + \ldots$$

Weyl-Wigner calculus!

Basic dictionary:

Divergences: $u \cdot \partial_x \star \varphi(x, u) = \partial_x^{\mu} \varphi_{\mu \ldots}(x) = \partial_x \cdot \partial_u \varphi(x, u)$

Contractions: $u_i \cdot u_j \star \varphi_i \varphi_j \to \partial_{u_i} \cdot \partial_{u_j} \varphi_i \varphi_j = \varphi_i^{\mu \ldots} \varphi_{j\,\mu \ldots}$

$(u_i \cdot u_j)^{\alpha} \star \varphi_i \varphi_j \to (\partial_{u_i} \cdot \partial_{u_j})^{\alpha} \varphi_i \varphi_j = \varphi_i^{\mu_1 \ldots \mu_\alpha \ldots} \varphi_{j\,\mu_1 \ldots \mu_\alpha \ldots}$

Ambient Space Formalism

Any d-dimensional constant curvature background can be embedded into a (d+1)-dimensional flat space!

$$x \to X \quad u \to U \qquad X^2 = L^2$$

Ambient space formulation:
Fronsdal 1979; Metsaev 1995; Biswas and Siegel 2002;
Bekaert, Buchbinder, Pashnev and Tsulaia 2004;
Hallowell and Waldron 2005;
Alkalaev, Barnich and Grigoriev 2006;
Francia, Mourad and Sagnotti 2008;
Boulanger, Iazeolla and Sundell 2009.

Formal simplification: non commutative nature of covariant derivatives disappear when using the ambient space language

General isomorphism between ambient space fields and (A)dS fields

$$X \cdot \partial_U \, \Phi(X, U) \; = \; 0$$

$$(X \cdot \partial_X \underbrace{- \, U \cdot \partial_U + 2 + \mu)}_{\Delta_h} \, \Phi(X, U) = 0$$

Flat space recovered in the L going to infinity limit: $X^M \to X^M + L\hat{N}^M$

$$(\hat{N} \cdot \partial_X - M)\Phi(X, U) = 0 \qquad \hat{N} \cdot \partial_U \Phi(X, U) = 0$$

Ambient space description of d-dimensional fields puts constant curvature backgrounds on "similar" footings!

Ambient Space Formalism

Gauge invariance compatibility with the tangentiality constraint fixes **μ** and puts constraints on the gauge parameters

$$\delta^{(0)}\Phi(X,\,U)\;=\;U\cdot\partial_X\,E(X,\,U)$$

$$\begin{cases} \mu\notin\mathbb{N} & E(X,U)=0 \\ \mu=r\in\mathbb{N} & E(X,U)=(U\cdot\partial_X)^r\,\Omega(X,U) \end{cases}$$

No gauge symmetry allowed!

$$X\cdot\partial_U\,\Omega(X,U)=0$$

r>0: partially-massless points (unitary only in dS) [S. Deser, R.I. Nepomechie, E. Waldron]

What about action principles??

We want just an ambient space description of d-dimensional physics...

$$\int d^{d+1}X\,\delta\left(\sqrt{X^{\,2}}\;-\;L\right)\;=\;\int_{(A)dS_D} d^dx\sqrt{-g}$$

Furthermore: we avoid problems coming from the diverging radial integrals that can spoil gauge invariance (ambient space total derivatives encode lower derivative terms when rewritten in intrinsic notation)

Total derivatives are not anymore zero!

$$\delta^{(n)}(R-L)\;=\;\delta(R-L)(\hat{\delta})^n \longrightarrow K_3(L^{-1},U_i,\partial_{X_i})\to K_3(\hat{\delta},U_i,\partial_{X_i})$$

Massless Interactions

$$\mathcal{L}^{(2)} \sim \frac{1}{2}\,\delta \sum \Phi_{M_1\ldots M_s}\,\partial_X^2\,\Phi^{M_1\ldots M_s} \;\rightarrow\; \mathcal{K}_2 \sim \frac{1}{2}\,\delta\, e^{U_1\cdot U_2}\,\partial_{X_2}^2$$

$$\delta^{(1)}\mathcal{L}^{(2)} + \delta^{(0)}\mathcal{L}^{(3)} = 0 \longrightarrow \partial_{X_i}\cdot\partial_{U_i}\,\mathcal{K}_3(U_1,\,U_2,\,U_3;\partial_{X_j}) \approx 0$$

$$\delta^{(0)} \sim \partial_X\cdot\partial_U$$

hep-th/1107.5843: M.T.

The cubic coupling Generating Function is the solution of linear homogeneous differential equations

Convenient simplification: the solution is a function of simple building blocks...

$$\left.\begin{aligned} Z_i &= U_{i-1}\cdot U_{i+1} \\ Y_i &= U_i\cdot\partial_{X_{i+1}} \end{aligned}\right\} \quad \left[Y_{i-1}\partial_{Z_{i+1}} - Y_{i+1}\partial_{Z_{i-1}} + \frac{\hat{\delta}}{L}\left(Y_{i-1}\partial_{Y_{i-1}} - Y_{i+1}\partial_{Y_{i+1}}\right)\partial_{Y_i}\right] K_3(\hat{\delta},\,Y,\,Z) = 0$$

hep-th/1110.5918: E. Joung and M.T.

The solution to this differential equation can be written reabsorbing the lower derivative terms into total derivatives as:

$$\mathcal{K}_3 = f\left(\tilde{Y}_1,\,\tilde{Y}_2,\,\tilde{Y}_3,\,\tilde{G}_{123}\right)$$

$$\tilde{Y}_i \equiv U_i\cdot\partial_{X_{i+1}} + \alpha_i\,U_i\cdot\partial_X$$

$$\tilde{G} \equiv (U_1\cdot\partial_{X_2} + \beta_1\,U_1\cdot\partial_X)\,U_2\cdot U_3 + (U_2\cdot\partial_{X_3} + \beta_2\,U_2\cdot\partial_X)\,U_3\cdot U_1 + (U_3\cdot\partial_{X_1} + \beta_3\,U_3\cdot\partial_X)\,U_1\cdot U_2$$

Related work in the frame-like approach: hep-th/1108.5921 M. Vasiliev

Flat limit

In the flat limit the total derivative terms simply vanish
and one recovers the simpler flat space couplings

The result is:

$$g_{123} = y_1 z_1 + y_2 z_2 + y_3 z_3$$

$$y_i = u_i \cdot \partial_{x_{i+1}} \qquad z_i = u_{i-1} \cdot u_{i+1}$$

String Theory coupling function (maybe not the only consistent choice...)

$$\mathcal{L}^{(3)} = \exp\left\{\sqrt{2\,\alpha'}\left[y_1 + y_2 + y_3 + g_{123}\right]\right\} \star_{123} \varphi_1(x_1; u_1)\, \varphi_2(x_2; u_2)\, \varphi_3(x_3; u_3)\Big|_{x_i = x}$$

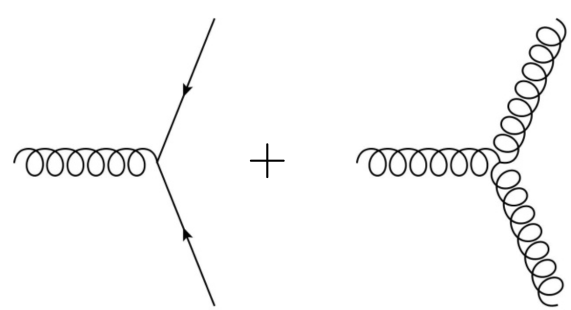
arXiv:1003.2877: Manvelyan, Mkrtchyan, Ruehl; hep-th/1006.5242: A.Sagnotti and M.T.

Completion uniquely fixed up to partial integrations and field redefinitions.

Non-abelian deformation of the gauge symmetry!

(reproduce exactly Metsaev's list in a covariant form)

$g_{123} =$



Lego bricks of any cubic coupling!

(A)dS and flat cubic vertices

Cubic (A)dS HS couplings can be encoded within the flat ones modulo a fixed boundary term!

$$\mathcal{K}_3^{(\mathrm{A})\mathrm{dS}} = \exp\left(G_{123}^{\,\mathrm{flat}}\right)\left[1 + \partial_X(\ldots) + \partial_X^2(\ldots) + \ldots\right] = \exp\left(G_{123}^{\,(\mathrm{A})\mathrm{dS}}\right)$$

$$G_{123}^{\,(A)dS} = G_{123}^{\,\mathrm{flat}} + U_1\cdot\partial_X\left(\alpha_1 + \beta_1\, U_2\cdot U_3\right) + \ldots$$

Lower derivative tail appears automatically in terms of intrinsic (A)dS coordinates after the reduction:

$$X^M = R\,\hat{X}^M(x)$$

The minimal coupling is recovered in (A)dS within the non-minimal ambient space coupling and requires by consistency a whole tail of higher derivative contributions:

$$\mathcal{L}^{(3)} \simeq A_0 + \frac{1}{\Lambda}A_2 + \ldots + \frac{1}{\Lambda^n}A_{2n}$$

This kind of approach may give some insights into Vasiliev's system!

Compute Vasiliev's system coupling function (work in progress...)

Also Massive couplings…

The couplings are still functions of simple building blocks and are solutions to a differential equation: (hep-th/1203.6578: E. Joung, L. Lopez and M.T.)

$$\left[Y_{i-1}\partial_{Z_{i+1}} - Y_{i+1}\partial_{Z_{i-1}} + \frac{\hat{\delta}}{L}\left(Y_{i-1}\partial_{Y_{i-1}} - Y_{i+1}\partial_{Y_{i+1}} - \frac{\mu_{i-1}-\mu_{i+1}}{2}\right)\partial_{Y_i}\right] K_3(\hat{\delta},\, Y,\, Z) \;=\; 0$$

3 massive $\mathcal{K}_3(Y_1, Y_2, Y_3, Z_1, Z_2, Z_3)$

1 massless and 2 massive (equal masses) $\mathcal{K}_3(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, Z_1, \tilde{G})$

1 massless and 2 massive (different masses) $\mathcal{K}_3(Y_2, Y_3, Z_1, \tilde{H}_2, \tilde{H}_3)$

2 massless and 1 massive $\mathcal{K}_3(Y_3, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$

$$\tilde{G} \equiv (U_1\cdot\partial_{X_2} + \beta_1\, U_1\cdot\partial_X)\, U_2\cdot U_3 + (U_2\cdot\partial_{X_3} + \beta_2\, U_2\cdot\partial_X)\, U_3\cdot U_1 + (U_3\cdot\partial_{X_1} + \beta_3\, U_3\cdot\partial_X)\, U_1\cdot U_2$$

$$\tilde{H}_i \equiv \partial_{X_{i+1}}\cdot\partial_{X_{i-1}}\, U_{i-1}\cdot U_{i+1} - \partial_{X_{i+1}}\cdot U_{i-1}\, \partial_{X_{i-1}}\cdot U_{i+1} \qquad Z_i \equiv U_{i+1}\cdot U_{i-1}$$

Flat space result in the L going to infinity limit!
Again YM gives the right building blocks!

$$\tilde{Y}_i \equiv U_i\cdot\partial_{X_{i+1}} + \alpha_i\, U_i\cdot\partial_X$$

...and Partially-massless ones

$$\left.\begin{array}{l}\mu = r \in \mathbb{N} \\ (X\cdot\partial_X - U\cdot\partial_U + 2 + \mu)\,\Phi(X,U) = 0\end{array}\right\} \quad \begin{array}{c}\delta\Phi(X,U) \;=\; (U\cdot\partial_X)^{r+1}\,\Omega(X,U) \\ X\cdot\partial_U\,\Omega(X,U) \;=\; 0\end{array}$$

Also these couplings are solutions of a linear homogeneous differential equation but it is not easy to find the generating function solution in general!

(hep-th/1203.6578: E. Joung, L. Lopez and M.T.)

A simplification arise since the equation factorizes as:

$$\prod_{k=0}^{r_1}\left[Y_3\partial_{Z_2} - Y_2\partial_{Z_3} + \tfrac{\hat{\delta}}{L}\left(Y_3\partial_{Y_3} - Y_2\partial_{Y_2} + \tfrac{r_1+\mu_2-\mu_3-2k}{2}\right)\partial_{Y_1}\right]\, K_3(\hat{\delta}, Y_i, Z_i) = 0$$

Together with permutations depending on the number of partially-massless fields

G and Y building blocks solving the massless equations appear in analogy with the one massless two massive case whenever

$$\mu_i - |\mu_{i+1} - \mu_{i-1}| \in 2\,\mathbb{N}$$

Otherwise H-building blocks are present!

Difficulty: Extract the polynomial solutions out of the general solution of the PDE

Work in progress...

For example: 4-4-2 couplings with the spin 4 at its first partially-massless point one gets 6 couplings (mathematica implementation)

$$
\begin{aligned}
\mathcal{K}^{(1)} &= Y_1^4\,Y_2^4\,Y_3^2 - 12\,\hat\delta^2\,Y_1^2\,Y_2^2\,(Y_1\,Z_1+Y_2\,Z_2)^2 + 48\,\hat\delta^3\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3) \\
&\quad - 24\,\hat\delta^4\,Z_3^2\left[6\,Y_1^2\,Z_1^2+6\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+2\,Y_1\,Z_1\,(7\,Y_2\,Z_2+2\,Y_3\,Z_3)\right]+96\,\hat\delta^5\,Z_1\,Z_2\,Z_3^3\,, \\
\mathcal{K}^{(2)} &= Y_1^3\,Y_2^3\,Y_3^2\,Z_3 - 3\,\hat\delta\,Y_1^2\,Y_2^2\,(Y_1\,Z_1+Y_2\,Z_2)^2 + 12\,\hat\delta^2\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3) \\
&\quad - 6\,\hat\delta^3\,Z_3^2\left[6\,Y_1^2\,Z_1^2+6\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+2\,Y_1\,Z_1\,(7\,Y_2\,Z_2+2\,Y_3\,Z_3)\right]+24\,\hat\delta^4\,Z_1\,Z_2\,Z_3^3\,, \\
\mathcal{K}^{(3)} &= Y_1^3\,Y_2^3\,Y_3\,(Y_1\,Z_1+Y_2\,Z_2) + \hat\delta\,Y_1^2Y_2^2\left(6\,Y_1^2\,Z_1^2+11\,Y_1\,Y_2\,Z_1\,Z_2+6\,Y_2^2\,Z_2^2\right) \\
&\quad -18\,\hat\delta^2\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3) \\
&\quad + 6\,\hat\delta^3\,Z_3^2\left[6\,Y_1^2\,Z_1^2+2\,Y_2\,Z_2\,(3\,Y_2\,Z_2+Y_3\,Z_3)+Y_1\,Z_1\,(15\,Y_2\,Z_2+2\,Y_3\,Z_3)\right]-12\,\hat\delta^4\,Z_1\,Z_2\,Z_3^3\,, \\
\mathcal{K}^{(4)} &= -\,Y_1^2\,Y_2^2\left(Y_1^2\,Z_1^2+2\,Y_1\,Y_2\,Z_1\,Z_2+Y_2^2\,Z_2^2-Y_3^2\,Z_3^2\right) \\
&\quad + 4\,\hat\delta\,Y_1\,Y_2\,(Y_1\,Z_1+Y_2\,Z_2)\,Z_3\,(2\,Y_1\,Z_1+2\,Y_2\,Z_2+Y_3\,Z_3) \\
&\quad - 2\,\hat\delta^2\,Z_3^2\left[6\,Y_1^2\,Z_1^2+6\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+2\,Y_1\,Z_1\,(7\,Y_2\,Z_2+2\,Y_3\,Z_3)\right]+8\,\hat\delta^3\,Z_1\,Z_2\,Z_3^3\,, \\
\mathcal{K}^{(5)} &= Y_1^2\,Y_2^2\,(Y_1\,Z_1+Y_2\,Z_2)\,(Y_1\,Z_1+Y_2\,Z_2+Y_3\,Z_3) \\
&\quad - \hat\delta\,Y_1\,Y_2\,Z_3\left[6\,Y_1^2\,Z_1^2+2\,Y_2\,Z_2\,(3\,Y_2\,Z_2+2\,Y_3\,Z_3)+Y_1\,Z_1\,(13\,Y_2\,Z_2+4\,Y_3\,Z_3)\right] \\
&\quad + 2\,\hat\delta^2\,Z_3^2\left[3\,Y_1^2\,Z_1^2+Y_2\,Z_2\,(3\,Y_2\,Z_2+Y_3\,Z_3)+Y_1\,Z_1\,(8\,Y_2\,Z_2+Y_3\,Z_3)\right]-2\,\hat\delta^3\,Z_1\,Z_2\,Z_3^3\,, \\
\mathcal{K}^{(6)} &= Y_1\,Y_2\,Z_3\,(Y_1\,Z_1+Y_2\,Z_2+Y_3\,Z_3)^2 \\
&\quad - \hat\delta\,Z_3^2\left[3\,Y_1^2\,Z_1^2+3\,Y_2^2\,Z_2^2+4\,Y_2\,Y_3\,Z_2\,Z_3+Y_3^2\,Z_3^2+4\,Y_1\,Z_1\,(2\,Y_2\,Z_2+Y_3\,Z_3)\right] \\
&\quad + 4\,\hat\delta^2\,Z_1\,Z_2\,Z_3^3\,.
\end{aligned}
$$

Although look complicated they are related to the same building blocks present for massless and massive fields

Can we go beyond?

$$\boxed{\delta^{(1)}\mathcal{L}^{(2)}} + \delta^{(0)}\mathcal{L}^{(3)} = 0$$

$$\boxed{\delta^{(1)}\mathcal{K}_3 \sim \delta^{(0)}\left(\mathcal{K}_3\,\frac{1}{\Box}\,\mathcal{K}_3\right)}$$

$$\delta^{(2)}\mathcal{L}^{(2)} + \delta^{(1)}\mathcal{L}^{(3)} + \delta^{(0)}\mathcal{L}^{(4)} = 0 \longrightarrow \partial_{X_i}\cdot\partial_{U_i}\,\mathcal{K}_4(U_1,\,\ldots) \approx \boxed{-\,\delta^{(1)}\mathcal{K}_3}$$

In general: not easy to address this problem…

Starting from the quartic order the differential equation is non-homogeneous

…but there is a logic!

$$\partial_{X_i}\cdot\partial_{U_i}\,\mathcal{K}_4^{\text{homo}}(U_1,\,\ldots) \approx 0$$

Split it into non-local contributions
(leave locality aside for a moment)

$$\mathcal{K}_4 = \boxed{\mathcal{K}_4^{\text{part}}} + \boxed{\mathcal{K}_4^{\text{homo}}}$$

$$\partial_{X_i}\cdot\partial_{U_i}\,\mathcal{K}_4^{\text{part}}(U_1,\,\ldots) \approx -\,\delta^{(1)}\mathcal{K}_3$$

> The particular solution to the non-homogeneous equation is always given by minus the current exchange contribution and is entirely specified by the lower-point couplings!

The Noether procedure is reduced to linear homogeneous differential equations (WARD IDENTITIES)

hep-th/1107.5843: M.T.

We can characterize contact Lagrangian quartic couplings as the counterterms compensating the violation of the linearized gauge invariance of the current exchange part

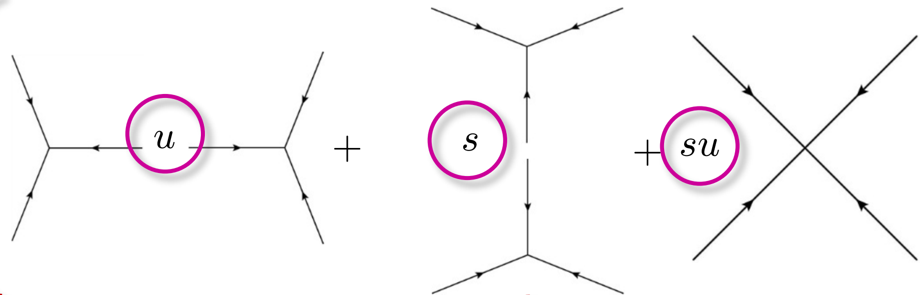
HS four-point functions

Solving for simplicity only the order zero piece of the differential equation in 1/L one gets the flat space solutions that can be deformed to constant curvature background afterwards

hep-th/1107.5843: M.T.

$$\partial_{X_i} \cdot \partial_{U_i} \mathcal{K}_4^{\text{homo}}(U_i; \partial_{X_i}) \approx 0$$

$$\mathcal{K}_4^{\text{homo}}(U_i; \partial_{X_i}) = \frac{1}{su} \exp \Big[\quad \Big]$$



$$su\, \mathcal{G}_{1234}(\partial_{X_i}, U_i)$$

Key point: Its power expansion reproduces the planar current exchanges of HS fields

Not only: also non-planar options are available:

$$\mathcal{K}_4^{\text{homo}}(U_i; \partial_{X_i}) = \frac{1}{stu} \exp\left(su\, G_{1234} + st\, G_{1243}\right)$$

First step in order to find the solution in any constant curvature background

To reiterate: similar spirit of the cubic case but one reconstructs the gauge invariant completion of the current exchanges recognizing the proper LEGO BRICKS (YM)!

Question: what about Lagrangian quartic couplings? Can they be local?

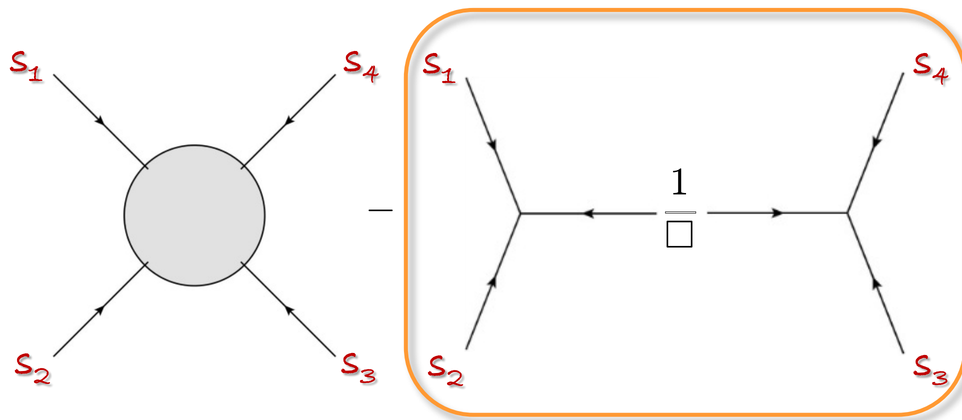
Non-localities?

From the S-matrix perspective everything seems standard...

...but if we extract the Lagrangian couplings explicit non-localities can arise!

hep-th/9304057: Barnich and Henneoux

$\mathcal{K}_4 =$



The Noether procedure requires the full current exchange! (depends on the cubic coupling function)

(A)dS may behave similarly:

$$\left(\frac{1}{\Box}\right)_{d+1} = \frac{1}{\Box - \Lambda} = -\frac{1}{\Lambda}\sum\left(\frac{\Box}{\Lambda}\right)^n$$

Non local 4-p couplings If the first term cannot factorize on all available exchanges!

(...unitarity??)

NON-LOCAL Geometry?? (Crucial to complete the analysis in (A)dS)

What is the alternative to locality at the Lagrangian level, if any?

...moreover this kind of structure forces infinitely many spins propagating!

Very difficult to define an S-matrix for infinitely many massless particles...

Outlook

- Three-point couplings in the ambient space formalism
- Also massive and partially-massless couplings
- First step beyond cubic: flat-space lego bricks (to be deformed in (A)dS !)
- Difficulty: understand the S-matrix for infinitely many massless particles

BEHIND THE CORNER:

Quartic couplings in constant curvature backgrounds

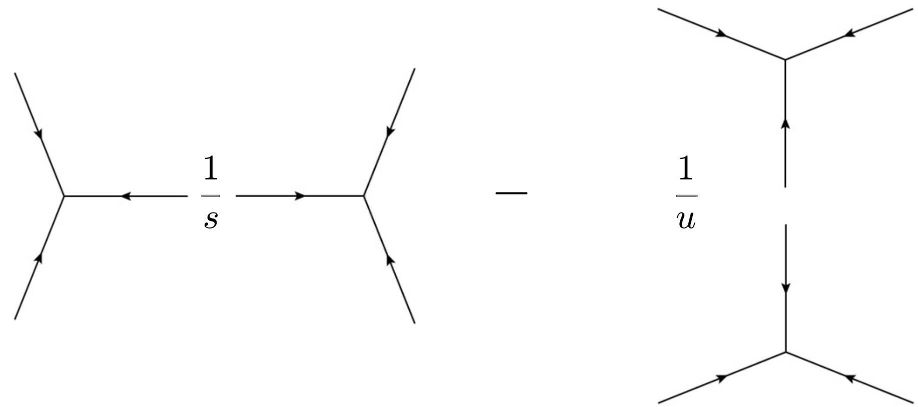
(work in progress...)

For the near future:

- Gauge algebra deformations and global symmetries
- Compute the coupling function of Vasiliev's system
- AdS/CFT applications
- This approach can be particularly suited in order to shed some light on String Theory on (A)dS

Warm up: Yang-Mills

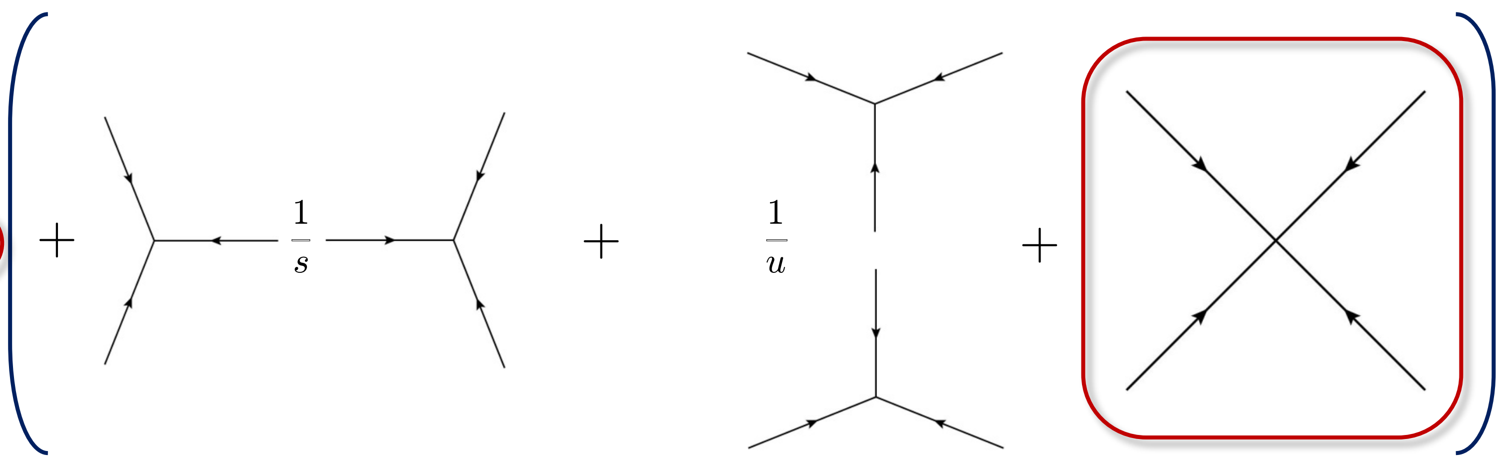
$\mathcal{L}^{(4)}_{\text{part}} = -$



"Color" global symmetry:
Color-ordered contribution,
only two channels

$\alpha = 1$

$\mathcal{L}^{(4)}_{\text{homo}} = \alpha$



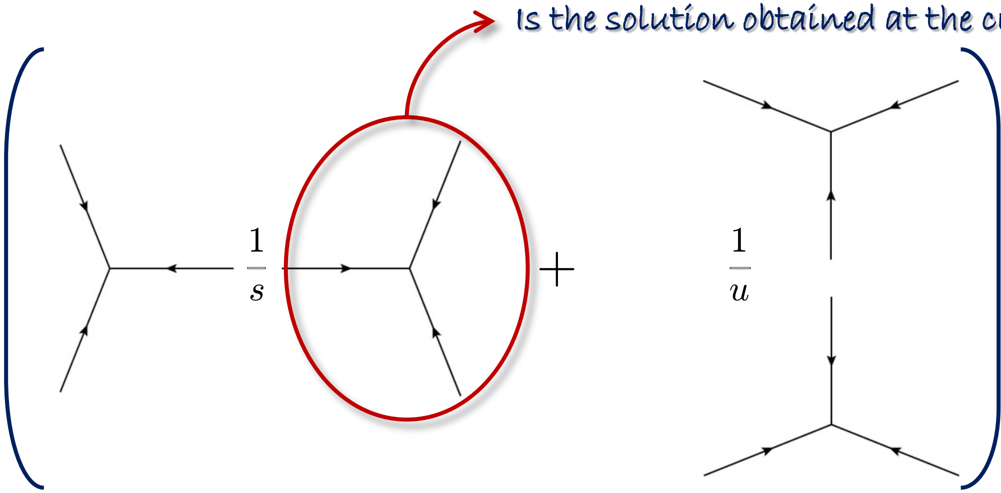
We can characterize contact Lagrangian quartic couplings as the counterterm compensating the violation of the linearized gauge invariance of the current exchange part

hep-th/1107.5843: M.T.

Warm up: Yang-Mills

Is the solution obtained at the cubic level

$\delta_4^{(0)}$



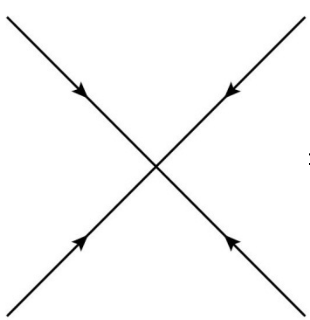
Non-abelian vertices require a non trivial mixing of different channels!

$$\partial_{X_4} \rightarrow U_4$$

$$= -2\,U_2\cdot\partial_{X_4} + U_1\cdot\partial_{X_4} + U_3\cdot\partial_{X_4} - 2\,U_1\cdot U_3\,U_2\cdot\partial_{X_4} + U_1\cdot U_2\,U_3\cdot\partial_{X_4} + U_2\cdot U_3\,U_1\cdot\partial_{X_4}$$

Two gauge bosons two scalars

Four gauge bosons



The procedure stops for YM: we recover the full 4-coupling in color-ordered form!

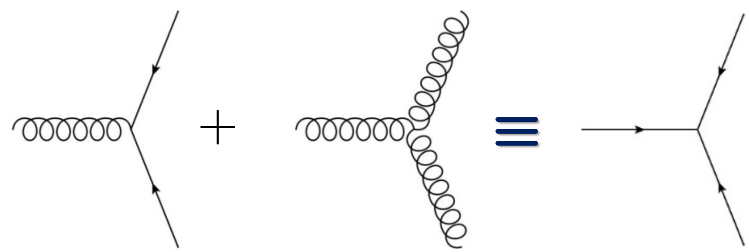
$$= 2\,U_1\cdot U_3\,U_2\cdot U_4 - U_1\cdot U_4\,U_2\cdot U_3 - U_1\cdot U_2\,U_3\cdot U_4$$
$$+ 2\,(U_1\cdot U_3 + U_2\cdot U_4) - U_1\cdot U_2 - U_2\cdot U_3 - U_3\cdot U_4 - U_4\cdot U_1$$

HS four-point functions

The cubic case suggests how to provide an answer to all orders:

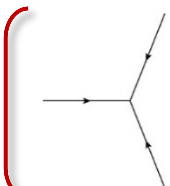
YM: Lego bricks of any scattering amplitude!

$G_{123} =$



Noether procedure indeed is solved by any generating function satisfying:

$$\partial_{X_i} \cdot \partial_{U_i} \, \mathcal{K}_3(U_1,\, U_2,\, U_3;\, \partial_{X_i}) \;\approx\; 0 \longrightarrow \mathcal{K}_3 \;=\; \exp \left(\quad \right)$$



Something similar is possible at the quartic order but at the level of amplitudes:

$$\partial_{X_i} \cdot \partial_{U_i} \, \mathcal{K}_4^{\text{homo}}(U_1,\, \ldots) \;\approx\; 0$$

Gauge Boson :

$$G_{1234}(\partial_{X_i}, U_i) \;=\; -\frac{1}{s}\, G_{12a} \star_a G_{a34} \;-\; \frac{1}{u}\, G_{41a} \star_a G_{a23} \;+\; V^{(4)}_{1234}$$

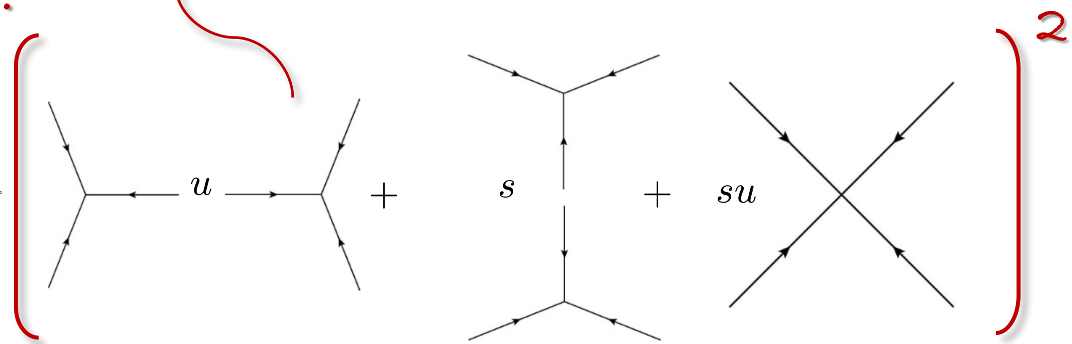
Colored spin-2 or Gravity?

Four spin-2 case: as we said two different kind of options!

Planar (open-string like):

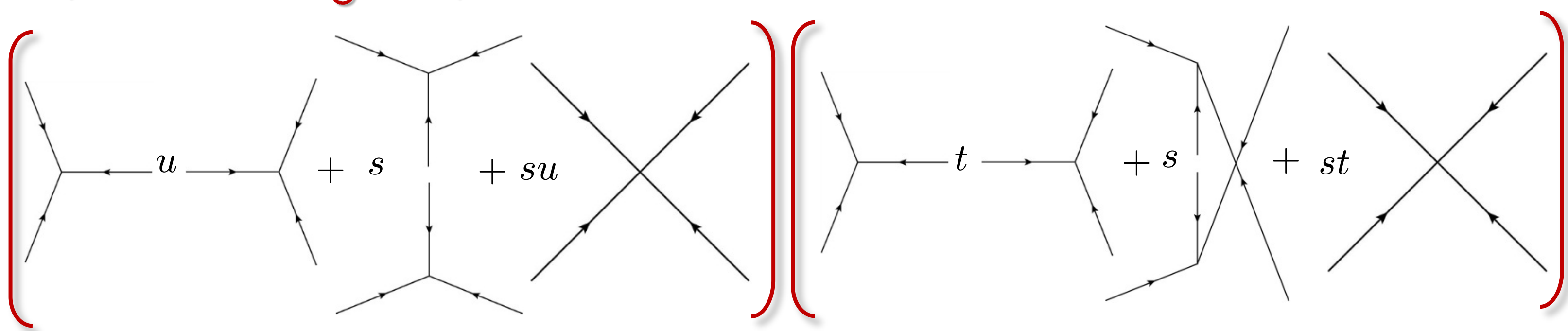
Just the four gauge boson amplitude squared

$$\mathcal{K}_4^{\text{homo}} = \frac{a(s,t,u)}{su}$$



Non-planar (closed-string like):

$$\mathcal{K}_4^{\text{homo}} = \frac{a(s,t,u)}{stu}$$



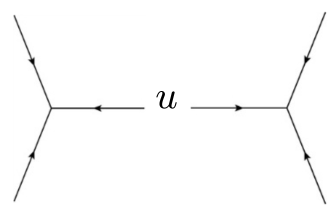
+ cyclic

Colored spin-2 or Gravity?

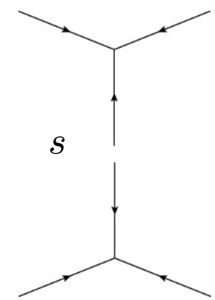
Let us expand the results focusing on the current exchange part!

Planar :

$\mathcal{K}_4^{\text{homo}} \sim \frac{1}{su} \Big($



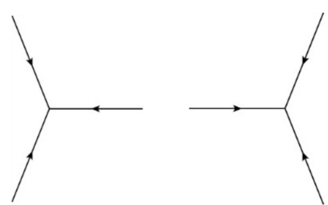
$\Big)^2 - \frac{1}{su} \Big($



$\Big)^2 + \ldots$

Non-planar:

$\mathcal{K}_4^{\text{homo}} \sim \frac{1}{s} \Big($



$\Big)^2 + \ldots$

Spin 3

Spin 2

Can we read the exchanged particles?

The second amplitude is the Gravitational one…

…but the first is not! (colored spin-2 requires HS)

Massless spin-2 not necessarily Gravity in HS theories
(mixing? See Vasiliev's system…)

hep-th/0701163: D. Francia, J. Mourad, A. Sagnotti

Off-shell HS Couplings

Strategy: Find counter terms proportional to traces and divergences ensuring gauge invariance up to full free Lagrangian Eom's

$$\mathcal{A}_{\pm} \sim \exp\left\{\sqrt{\frac{\alpha'}{2}}\,\mathcal{G}_{123}\right\}\left[1 + \frac{\alpha'}{2}\mathcal{H}_{12}\mathcal{H}_{13} + \left(\sqrt{\frac{\alpha'}{2}}\right)^3 \mathcal{H}_{21}\mathcal{H}_{32}\mathcal{H}_{13} + \text{Cyclic}\right]$$

$$\star_{123}\ \phi_1(x_1;u_1)\,\phi_2(x_2;u_2)\,\phi_3(x_3;u_3)$$

$$\mathcal{H}_{ij} = (\xi_i \cdot \xi_j + 1)\,i\mathcal{D}_j - p_j \cdot \xi_i\,\mathcal{A}_j$$

Generalized de Donder operator

$$\delta^{(0)}(\mathcal{D} \star \phi) = \Box\,\Lambda$$

A-operator (∼ trace, ...)

$$\delta^{(0)}(\mathcal{A} \star \phi) = -\partial_x \cdot \partial_u\,\Lambda$$

The physical content is encoded into G!

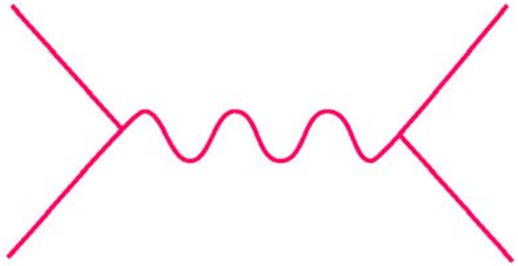
The same procedure works also at higher orders!

Current Exchanges

Only irreducible degrees of freedom propagate

$$\mathcal{P}^{(s)}_{\mu_1\cdots\mu_s;\nu_1\cdots\nu_s} = \frac{1}{p^2+M^2} P_{\mu_1\cdots\mu_s;\nu_1\cdots\nu_s}$$

Projector onto the traceless-transverse part

$\mathcal{J}$  $\mathcal{J}$

same trick: use the symbols

$$\mathcal{P}^{(s)}(\xi^2\zeta^2, \xi\cdot\zeta)$$

Homogeneous symmetric polynomial of degree m for totally symmetric tensors

Massless case: only traceless projector

Trace operator $\Longleftrightarrow$ $\partial_\xi \cdot \partial_\xi$

(Laplace Operator)

The propagator is associated to a harmonic polynomial

The result for massless totally symmetric fields is

$$\mathcal{P}^{(s)}(\xi,\zeta) = \frac{1}{p^2}\left\{K\ \left(\xi^2\zeta^2\right)^{s/2} f_s^{[\frac{d}{2}-2]}\left(\frac{\xi\cdot\zeta}{\sqrt{\xi^2\zeta^2}}\right)\right\}$$

$$f_s^{[\frac{d}{2}-2]}(x) = G_s^{[\frac{d}{2}-2]}(x) \quad d>4$$

$$f_s^{[\frac{d}{2}-2]}(x) = T_s(x) \quad d=4$$

Generating functions of exchanges

Generating functions sum an infinite number of propagators with arbitrary coupling constants

$$\hat{\mathcal{P}} = \frac{1}{p^2}\left[a\left(\frac{1}{2}\xi\cdot\zeta + \frac{1}{2}\sqrt{(\xi\cdot\zeta)^2 - \xi^2\zeta^2}\right) + a\left(\frac{1}{2}\xi\cdot\zeta - \frac{1}{2}\sqrt{(\xi\cdot\zeta)^2 - \xi^2\zeta^2}\right) - a_0\right]$$

Massless HS in D=4

$$\hat{\mathcal{P}} = \alpha'\int_0^1 d\lambda\,\lambda^{-\alpha' s-2}\left[a\left(\frac{\lambda}{2}\xi\cdot\zeta + \frac{\lambda}{2}\sqrt{(\xi\cdot\zeta)^2 - \xi^2\zeta^2}\right) + a\left(\frac{\lambda}{2}\xi\cdot\zeta - \frac{\lambda}{2}\sqrt{(\xi\cdot\zeta)^2 - \xi^2\zeta^2}\right) - a_0\right]$$

Massive HS in D=3 (First Regge trajectory multiplet)

The result for arbitrary D is complicated and simplifies only for some particular choices of the coupling constants

$$a(t) = \frac{1}{(1-t)^\alpha} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,t^n$$

$$\hat{\mathcal{P}} = \frac{1}{p^2}\left(1 - \xi\cdot\zeta + \frac{\xi^2\zeta^2}{4}\right)^{-\alpha}$$

Massless HS in D=2 α+4

$$\hat{\mathcal{P}} = \alpha'\int_0^1 d\lambda\,\lambda^{-\alpha' s-2}\left(1 - \xi\cdot\zeta\,\lambda + \frac{\xi^2\zeta^2\,\lambda^2}{4}\right)^{-\alpha}$$

$$a(z) = \sum_{n=0}^{\infty}\frac{1}{n!}a_n z^n$$

Coupling function (Bekaert, Joung, Mourad, 2009)

Massive HS in D=2 α+3 (First Regge trajectory multiplet)

Scattering Amplitudes

The limiting currents can be used to compute scattering amplitudes with infinitely many higher-spin particle exchanges

$$J_\pm = \exp\left(\pm\sqrt{\frac{\alpha'}{2}}p_{12}\cdot k\right)\phi_1\left(p_1, \pm\sqrt{2\alpha'}p_2\right)\phi_2\left(p_2, \mp\sqrt{2\alpha'}p_1\right)$$

S-channel

$$\mathcal{A}^{(s)} = -\frac{1}{\alpha' s}\left[a\left(\frac{\alpha'}{4}(u-t)+\frac{\alpha'}{2}\sqrt{-ut}\right)+a\left(\frac{\alpha'}{4}(u-t)-\frac{\alpha'}{2}\sqrt{-ut}\right)-a_0\right]$$
$$\times\,\phi_1\left(k_1, \pm\sqrt{2\alpha'}p_2\right)\phi_2\left(k_2, \mp\sqrt{2\alpha'}p_1\right)\phi_3\left(k_1, \pm\sqrt{2\alpha'}p_4\right)\phi_4\left(k_2, \mp\sqrt{2\alpha'}p_3\right)$$

Massless HS in D=4

$$\mathcal{A}^{(s)} = \alpha'\int_0^1 d\lambda\,\lambda^{-\alpha' s-2}\left[a\left(\frac{\alpha'\lambda}{4}(u-t)+\frac{\alpha'\lambda}{2}\sqrt{-ut}\right)+a\left(\frac{\alpha'\lambda}{4}(u-t)-\frac{\alpha'\lambda}{2}\sqrt{-ut}\right)-a_0\right]$$
$$\times\,\phi_1\left(p_1, \pm\sqrt{2\alpha'}p_2\right)\phi_2\left(p_2, \mp\sqrt{2\alpha'}p_1\right)\phi_3\left(p_1, \pm\sqrt{2\alpha'}p_4\right)\phi_4\left(p_2, \mp\sqrt{2\alpha'}p_3\right)$$

"String" Spectrum in D=3

Same structure for the other currents, in higher dimensions and for massive particles

Scattering Amplitudes

For some choice of the coupling function the amplitude develops singularities in Mandelstam variables: form factor?

$$a(z) = \frac{1}{1-z}$$

For other choices the amplitude is well behaved at high energies but this property is lost after crossing

$$a(z) = e^z$$

In higher dimensions for some coupling functions the amplitude develops also cuts

$$a(z) = \frac{1}{(1-z)^\alpha}$$

The kinematical singularities appear to be a sign of non-local objects!
Non-local quartic couplings can be an option

(Open) Bosonic-String S-matrix

Chan-Paton factors

Gauge fixed version of the Polyakov path integral

$$S^{open}_{j_1\cdots j_n} = \int_{\mathbb{R}^{n-3}} dy_4 \cdots dy_n \, |y_{12}y_{13}y_{23}|$$
$$\times \langle \mathcal{V}_{j_1}(\hat{y}_1)\mathcal{V}_{j_2}(\hat{y}_2)\mathcal{V}_{j_3}(\hat{y}_3)\cdots\mathcal{V}_{j_n}(y_n)\rangle Tr(\Lambda^{a_1}\cdots\Lambda^{a_n}) + (1 \leftrightarrow 2)$$

$y_{ij} = y_i - y_j$

Vertex operators associated to asymptotic states via the state-operator isomorphism

$$(L_0 - 1)\,|\phi\rangle = 0 \qquad L_1\,|\phi\rangle = 0 \qquad L_2\,|\phi\rangle = 0$$

Generalized form of the Fierz-Pauli equations for massive fields!

$$(\Box - m_s^2)\,\phi_{\mu_1\ldots\mu_s} = 0 \qquad \partial^{\mu_1}\phi_{\mu_1\ldots\mu_s} = 0 \qquad \phi^{\alpha}{}_{\alpha\mu_3\ldots\mu_s} = 0$$

Three-point Amplitudes

Eliminate the unphysical dependence imposing the Virasoro constraints at the level of the generating function Z

$$-p_1^2 = \frac{s_1 - 1}{\alpha'} \qquad -p_2^2 = \frac{s_2 - 1}{\alpha'} \qquad -p_3^2 = \frac{s_3 - 1}{\alpha'} \qquad p_i \cdot \xi_i = 0 \qquad \xi_i \cdot \xi_i = 0$$

As in any S-matrix theory one recover on-shell results (Geometry is hidden...)

$$\mathbf{Z}_{phys} \sim exp\left\{\sqrt{\frac{\alpha'}{2}}\left(\xi_1 \cdot p_{23}\left\langle\frac{y_{23}}{y_{12}y_{13}}\right\rangle + \xi_2 \cdot p_{31}\left\langle\frac{y_{13}}{y_{12}y_{23}}\right\rangle + \xi_3 \cdot p_{12}\left\langle\frac{y_{12}}{y_{13}y_{23}}\right\rangle\right) + (\xi_1 \cdot \xi_2 + \xi_1 \cdot \xi_3 + \xi_2 \cdot \xi_3)\right\}$$

Just a sign!

$$p_{ij} = p_i - p_j$$

On-shell Couplings: Star product with generating functions of fields

$$\mathcal{A} = \phi_1\left(p_1, \frac{\partial}{\partial\xi} + \sqrt{\frac{\alpha'}{2}}\, p_{31}\right) \phi_2\left(p_2, \xi + \frac{\partial}{\partial\xi} + \sqrt{\frac{\alpha'}{2}}\, p_{23}\right) \phi_3\left(p_3, \xi + \sqrt{\frac{\alpha'}{2}}\, p_{12}\right)\Bigg|_{\xi=0}$$

Master Thesis (2009) [hep-th/1005.3061]

Similar results for four-point amplitudes (also Superstring and Mixed-symmetry fields in progress)

Examples

$$p_{ij} = p_i - p_j$$

0-0-s:

$$\mathcal{A}_{0-0-s} = \left(\sqrt{\frac{\alpha'}{2}}\right)^s \phi_1(p_1)\,\phi_2(p_2)\,\phi_3(p_3)\cdot p_{12}^s$$

Induced by a conserved current:

$$J(x,\xi) = \Phi\left(x + i\sqrt{\frac{\alpha'}{2}}\,\xi\right)\,\Phi\left(x - i\sqrt{\frac{\alpha'}{2}}\,\xi\right)$$

(Berends, Burgers and Van Dam, 1986; Bekaert, Joung, Mourad, 2009)

The complex scalar

$$= \phi(x)^*\phi(x) + i\sqrt{\frac{\alpha'}{2}}\,\xi^\mu\left[\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x)\right] + \ldots$$

HS currents

1-1-s:

$$\mathcal{A}_{s-1-1} = \left(\sqrt{\frac{\alpha'}{2}}\right)^{s-2} s(s-1)\,A_{1\mu}\,A_{2\nu}\,\phi^{\mu\nu\cdots}p_{12}^{s-2}$$
$$+ \left(\sqrt{\frac{\alpha'}{2}}\right)^{s}\left[A_1\cdot A_2\,\phi\cdot p_{12}^s + sA_1\cdot p_{23}\,A_{2\,\nu}\,\phi^{\nu\cdots}p_{12}^{s-1} + sA_2\cdot p_{31}\,A_{1\,\nu}\,\phi^{\nu\cdots}p_{12}^{s-1}\right]$$
$$+ \left(\sqrt{\frac{\alpha'}{2}}\right)^{s+2} A_1\cdot p_{23}\,A_2\cdot p_{31}\,\phi\cdot p_{12}^s$$

The amplitudes can contain extra "stuff" that drops out in the massless limit where genuine Noether interactions ought to be recovered! Similar to a scaling limit

This coupling too is induced by a conserved current!