New Families of the Knizhnik-Zamolodchikov-Bernard Equations Related to the WZW Models

Ginzburg Conference of Physics
Lebedev Institute, Moscow

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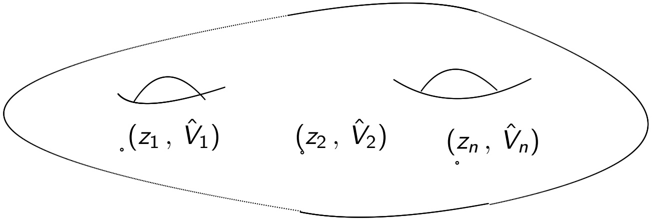
May 27, 2012

PRELIMINARIES

WZW theory on $\Sigma_{g,n}$



1. G-complex simple Lie group
2. $\Sigma_{g,n}$ Riemann surface of genus g with n marked points,
3. $\hat{V}_{\mu_1}, \ldots, \hat{V}_{\mu_n}$ integrable representations attached to the marked points.

$\Phi_a \in \hat{V}_a \otimes \hat{V}_a^*$, $\langle \Phi_1, \ldots, \Phi_n \rangle$ - correlators in WZW theory.

KZB eqauations

- Moving points:

$$\nabla_{z_a}\langle\Phi_1,\ldots,\Phi_n\rangle = 0\,,\ \boxed{\nabla_{z_a} = \kappa\partial_{z_a} + \hat{H}_a}\,,$$

•• Moduli of curves ($g > 0$)

$$\nabla_{\tau_j}\langle\Phi_1,\ldots,\Phi_n\rangle = 0\,,\ \boxed{\nabla_{\tau_j} = \kappa\partial_{\tau_j} + \hat{H}_{\tau_j}}\,.$$

$a = 1,\ldots,n$, $j = 1,\ldots,3g-3$, $\kappa = k + h^\vee$, $h^\vee$ - dual Coxeter number.

Flatness:

$$[\nabla_{z_a},\nabla_{z_b}] = 0\,,\ [\nabla_{\tau_j},\nabla_{\tau_k}] = 0\,,\ [\nabla_{z_a},\nabla_{\tau_k}] = 0$$

APPLICATIONS
i. Classical and quantum integrable systems

$$\boxed{\text{KZB equation}} \xrightarrow{\kappa\to 0} \boxed{\text{Quantum Hitchin Systems}}$$

$$\downarrow \hbar \to 0 \qquad\qquad \downarrow \hbar \to 0$$

$$\boxed{\text{Isomonodromy problem}} \xrightarrow{\kappa\to 0} \boxed{\text{Classical Hitchin Systems}}$$

Isomonodromy problem: Painleve type equations, Schlesinger systems...
Hitchin Systems: Calogero-Moser systems, Toda system, integrable Euler -Arnold tops...

ii. Classical dynamical r-matrix

$$\nabla_{z_a} = \kappa\partial_{z_a} + \hat{H}_a\,,$$

$\hat{H}_a$ - classical dynamical r-matrix related to the Riemann surface Σ_g and the group G.

$$[\nabla_{z_a}, \nabla_{z_b}] = 0 \Leftrightarrow \text{Classical Dynamical}$$

$$\text{Yang} - \text{Baxter Equation}$$

Example: I. KZ-equation

t_α -basis in $\mathfrak{g} = Lie(G)$, $\Sigma_{0,n} = \mathbb{C}P^1$
$z_1, \dots, z_n \in \mathbb{C}P^1$ - marked points,
$\mathfrak{t}^1_\alpha \dots, \mathfrak{t}^n_\alpha$, $\mathfrak{t}^a$ operator of representation of complex simple algebra $\mathfrak{g}$ in $V_a \subset \hat{V}_a$;

F-conformal blocks in the G WZW theory -

$F(z_1, z_2, \dots, z_n) \in \bigotimes_{a=1}^n V_a^*$

$$\left(\kappa\partial_a + \sum_{c\neq a} \frac{t^a_\alpha \otimes t^c_\alpha}{z_a - z_b}\right) F = 0\,.$$

1. $\kappa \to 0$ Quantum Gaudin system
2. $\hbar \to 0$ Classical Schlesinger system

$(\hbar \to 0) \sim t^a \to S^a$, $S^a \in \mathcal{O}_a$ - coadjoint orbit.

$$r^{ac} = \sum_{c\neq a} \frac{t^a_\alpha \otimes t^c_\alpha}{z_a - z_b} \; - \; \text{classical rational } r\text{−matrix}$$

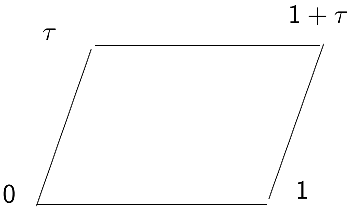
II. Bernard equation

$\Sigma_{1,1} = \Sigma_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$
V- spin l representation of $\mathrm{sl}(2,\mathbb{C})$

$\wp(x+1) = \wp(x+\tau) = \wp(x)$ -Weierschtrass function.

$$\left(\kappa\partial_\tau - \frac{1}{2}\hbar^2\partial_u^2 + l(l+1)\wp(2u)\right)F = 0\,.$$

$\kappa \to 0$ - *Elliptic Calogero-Moser system*;
$\hbar \to 0$ ($\hbar^2\partial_u^2 \to -p^2$, $\{p,u\} = 1$) - *Painleve 6 equation*;



- If the center $\mathcal{Z}(G)$ of G is non-trivial then there is ord$(\mathcal{Z}(G))$ different KZB equations

$$\nabla_j F_j = 0\,, \quad j = 1 \ldots ord(\mathcal{Z}(G))$$

- There is Hecke transformations (HT) of conformal blocks

$$HF_j = F_{j+1}$$

- HT is provided by monopoles in SUSY 4d Yang-Mills theory.

KZB equations on for SL$(2,\mathbb{C})$ on a torus $\Sigma_{\tau,n}$

$\mathcal{Z}(\mathrm{SL}(2,\mathbb{C})) = \mathbb{Z}_2 = \{s = (0,1)\}$

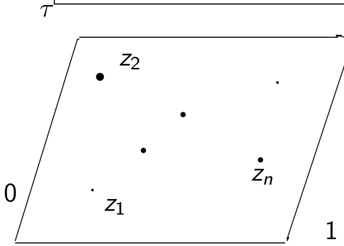
I. Trivial case $(s = 0)$ $G =$SL$(2,\mathbb{C})$

II. Non-trivial case $(s = 1)$

$G =$SL$(2,\mathbb{C})/$(center)$=$PSL$(2,\mathbb{C}) = G_{ad}$

$$\begin{cases} \nabla_a^{(s)} F = 0 \quad (a = 1,\dots,n) \\ \nabla_\tau^{(s)} F = 0 \end{cases} \qquad s = 0,1$$

$$[\nabla_a^{(s)}, \nabla_b^{(s)}] = 0\,, \quad [\nabla_a^{(s)}, \nabla_\tau^{(s)}] = 0$$



KZB equations on for SL$(2,\mathbb{C})$ on a torus $\Sigma_{\tau,n}$

I. Trivial case $s=0$

$$\nabla_a^{(0)} = \partial_{z_a} + \hat{\partial}_u^a + \sum_{c\neq a} r_0^{ac}$$

(e,h,f)- Chevalley basis in sl(2)

$$r_0^{ac} = E_1(z_a - z_c)h^a \otimes h^c + \quad (\mathrm{IRF\ r-matrix})$$

$$+2e^{2\pi i(z_a - z_c)}\phi\Big(2u+\tau, z_a - z_c\Big)(e^a \otimes f^c + f^a \otimes e^c)$$

$e^a = 1 \times \ldots 1 \otimes e \otimes 1 \ldots \otimes 1$ (e on the a-th place)

$\phi(u,z) = \frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)}$, $E_1(z) = \partial_z \ln\theta(z)$

$$\nabla_{\tau}^{(0)} = 2\pi i \partial_{\tau} + \frac{1}{2}\partial_u^2 + \frac{1}{2}\sum_{b,d} f_0^{bd} \,,$$

$n = 1$ - Bernard equation.

$$f_0^{ac} = (E_1^2(z_a - z_c) - \wp(z_a - z_c))h^a \otimes h^c +$$

$$+2e^{2\pi i(z_a - z_c)} f\Big(2u + \tau, z_a - z_c\Big)(e^a \otimes f^c + f^a \otimes e^c)$$

$f(z) = \partial_u \phi(u, z)$

$$[\nabla_a^{(0)}, \nabla_b^{(0)}] = 0 \ (a, b = 1, \ldots, n) \Leftrightarrow \text{CDYB eq.}$$

KZB equations on for $SL(2,\mathbb{C})$ on a torus $\Sigma_{\tau,n}$

I.Non-trivial case $s = 1$

$$\nabla_a^{(1)} = \partial_{z_a} + \sum_{c \neq a} r_1^{ac}$$

$$\nabla_\tau^{(1)} = 2\pi i \partial_\tau + + \frac{1}{2} \sum_{b,d} f_1^{bd} \,,$$

$$r_1^{ac} = e^{2\pi i (z_a - z_c)} \Big(\phi\Big(\tfrac{\tau}{2}, z_a - z_c\Big) \sigma_1^a \otimes \sigma_1^c + \phi\Big(\tfrac{\tau}{2} + \tfrac{1}{2}, z_a - z_c\Big) \sigma_2^a \otimes \sigma_2^c +$$

$$+\phi\Big(\tfrac{1}{2}, z_a - z_c\Big) \sigma_3^a \otimes \sigma_3^c \Big) \quad (\text{Belavin} - \text{Drinfeld r} - \text{matrix})$$

$$f_1^{ac} = e^{2\pi i (z_a - z_c)} \Big(f\Big(\tfrac{\tau}{2}, z_a - z_c\Big) \sigma_1^a \otimes \sigma_1^c + f\Big(\tfrac{\tau}{2} + \tfrac{1}{2}, z_a - z_c\Big) \sigma_2^a \otimes \sigma_2^c +$$

$$+f\Big(\tfrac{1}{2}, z_a - z_c\Big) \sigma_3^a \otimes \sigma_3^c \Big)$$

$$[\nabla_a^{(1)}, \nabla_b^{(1)}] = 0 \ (a, b = 1, \ldots, n) \Leftrightarrow \text{CYB eq.}$$

Center of Simple Lie groups

$G = \bar{G}$ - simply-connected complex simple Lie group ,

$G = G_{ad} = \bar{G}/\mathcal{Z}(\bar{G})$

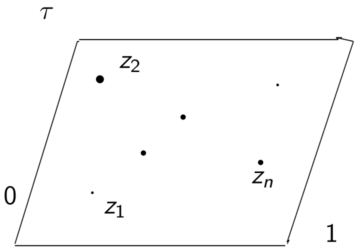
$\bar{G}$	Lie $(\bar{G})$	$\mathcal{Z}(\bar{G})$	G_{ad}
$SL(n, \mathbb{C})$	A_{n-1}	$\mathbb{Z}_n$	$SL(n, \mathbb{C})/\mathbb{Z}_n$
$Spin_{2n+1}(\mathbb{C})$	B_n	$\mathbb{Z}_2$	$SO(2n+1)$
$Sp_n(\mathbb{C})$	C_n	$\mathbb{Z}_2$	$Sp_n(\mathbb{C})/\mathbb{Z}_2$
$Spin_{4n}(\mathbb{C})$	D_{2n}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$SO(4n)/\mathbb{Z}_2$
$Spin_{4n+2}(\mathbb{C})$	D_{2n+1}	$\mathbb{Z}_4$	$SO(4n+2)/\mathbb{Z}_2$
$E_6(\mathbb{C})$	E_6	$\mathbb{Z}_3$	$E_6(\mathbb{C})/\mathbb{Z}_3$
$E_7(\mathbb{C})$	E_7	$\mathbb{Z}_2$	$E_7(\mathbb{C})/\mathbb{Z}_2$

WZW theory on a torus

$$X(z+1) = Ad_{\mathcal{Q}} X(z)\,,\;\; X(z+\tau) = Ad_{\Lambda} X(z)\,,$$

$$X \in \; Lie(G)$$

$$\boxed{\mathcal{Q}(z+\tau)\Lambda(z)\mathcal{Q}(z)^{-1}\Lambda^{-1}(z+1) = \zeta} \quad (*)\;\zeta \in \mathcal{Z}(G)$$



Moduli space $\mathcal{M}(G)$ = (solutions of $(*)$)/(conjugation)

$\mathcal{M}(G) = \{(\mathcal{Q}, \Lambda)\}$, $\mathcal{Q} \in H_G$ - Cartan subgroup, $\mathcal{Q} = \exp\left(2\pi i \frac{\rho^\vee}{h}\right)$ $\rho^\vee$ is a half-sum of positive coroots, h is the Coxeter number,

$$\Lambda = \Lambda_0 e^{2\pi i \mathbf{u}}\,, \ \ e^{2\pi i \mathbf{u}} \in \tilde{H}_0 \subset H\,, \ \ Ad_{\Lambda_0} e^{2\pi i \mathbf{u}} = e^{2\pi i \mathbf{u}}$$

$\mathbf{u}$ is an element of the moduli space $\mathcal{M}(G)$.
Λ_0 is an element of the Weyl group defined by ζ:

$$\zeta \to \Lambda_0(\zeta)\,, \quad \mathbf{u} = \mathbf{u}(\zeta)\,.$$

KZB connection for arbitrary G and $\zeta \in \mathcal{Z}(G)$

$$\nabla_a^{(\zeta)} F = 0\,, \;\; (a = 1, \ldots, n)\,, \quad \nabla_\tau^{(\zeta)} F = 0$$

$$\nabla_a^{(\zeta)} = \partial_{z_a} + \hat{\partial}_{\mathbf{u}}^{\;a} + \sum_{c \neq a} r^{ac}(\mathbf{u}, \zeta)\,,$$

$$\nabla_\tau^{(\zeta)} = 2\pi i \partial_\tau + \Delta_u + \frac{1}{2} \sum_{b,d} f^{bd}(\mathbf{u}, \zeta)\,,$$

$$r(\mathbf{u}, z) = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_\alpha^k(\mathbf{u}, z)\, \mathfrak{t}_\alpha^k \otimes \mathfrak{t}_{-\alpha}^{-k} + \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} \varphi_0^k(\mathbf{u}, z)\, \mathfrak{H}_\alpha^k \otimes \mathfrak{h}_\alpha^{-k}$$

$$f^{ac}(\zeta) = \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 f_\alpha^k(\mathbf{u}, z_a - z_c) \mathfrak{t}_\alpha^{k,a} \otimes \mathfrak{t}_{-\alpha}^{-k,c} + \sum_{k=0}^{l-1} \sum_{\alpha \in \Pi} f_0^k(\mathbf{u}, z_a - z_c) \mathfrak{H}_\alpha^{k,a} \otimes \mathfrak{h}_\alpha^{k,c}\,,$$

$$\mathfrak{t}_\alpha^{k,a} = 1 \otimes \ldots 1 \otimes \mathfrak{t}_\alpha^k \otimes 1 \ldots \otimes 1$$

Classical Systems and KZB $L^{(\zeta)} = L(\mathbf{u}, \zeta, z, \ldots)$,

$L^{(\zeta)} \in \mathfrak{g}$ -Lax operator in the classical isomonodromy problem

$$\partial_z + L^{(\zeta)}, \ \partial_{\bar{z}} + \bar{L}^{(\zeta)},$$

$$\mathcal{F}^{(\zeta)}_{z,\bar{z}} = [\partial_z + L^{(\zeta)}, \partial_{\bar{z}} + \bar{L}^{(\zeta)}] = 0$$

$$\boxed{L^{(\zeta)} \to (\nabla^{(\zeta)}_a, \ \nabla^{(\zeta)}_\tau)}$$

3d theory

$W = \Sigma \times \mathbb{R}\,,\ (z, \bar{z}, y),$
Field content:

$$A_z(z,\bar{z},y)_{y=-\infty} = L^{(\zeta_1)} \quad A_z(z,\bar{z},y)_{y=\infty} = L^{(\zeta_2)}$$
$$A_{\bar{z}}(z,\bar{z},y)_{y=-\infty} = \bar{L}^{(\zeta_1)} \quad A_{\bar{z}}(z,\bar{z},y)_{y=\infty} = \bar{L}^{(\zeta_2)}$$

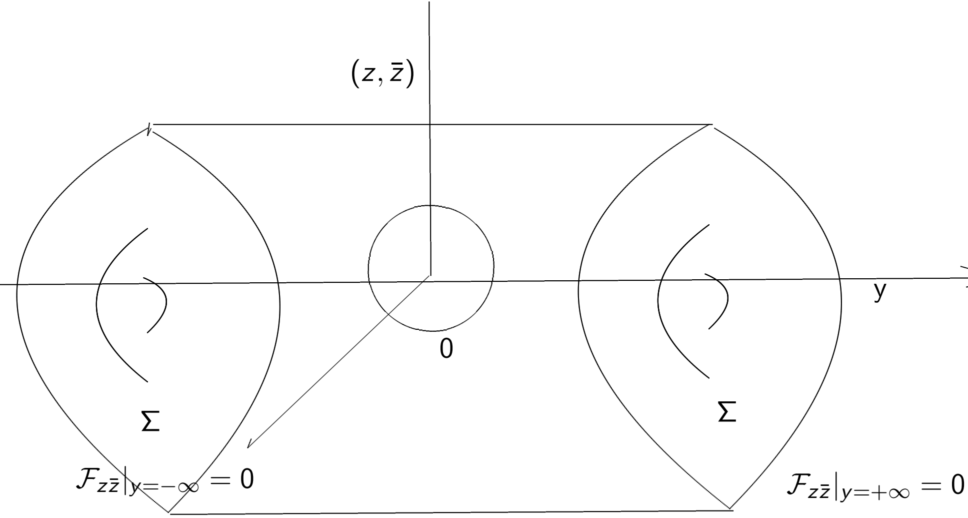
$$\partial_y + A_y(z,\bar{z},y)\,, \quad \phi(z,\bar{z},y) \in \mathfrak{g}$$

Bogomolny equation;

$$\boxed{F = *D\phi}\,, \quad * - \textit{Hodge operator in } W$$

Boundary conditions: $F_{z,\bar{z}}(z,\bar{z},y)|_{y=\pm\infty} = 0.$

3d theory



$(z, \bar{z})$
0
y
Σ
Σ
$\mathcal{F}_{z\bar{z}}|_{y=-\infty} = 0$
$\mathcal{F}_{z\bar{z}}|_{y=+\infty} = 0$

Bogomolny equation

$$\begin{cases} \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] = \frac{i}{2}(\partial_y \phi + [A_y, \phi]) \\ \partial_y A_z - \partial_z A_y + [A_z, A_y] = i(\partial_z \phi + [A_z, \phi]) \\ \partial_y A_{\bar{z}} - \partial_{\bar{z}} A_y + [A_{\bar{z}}, A_y] = -i(\partial_{\bar{z}} \phi + [A_{\bar{z}}, \phi]) \end{cases}$$

Boundary conditions: $F_{z,\bar{z}}|_{y=\pm\infty} = 0$.

$A_z|_{y=\pm\infty}$ corresponds to Lax operators $L_\pm$

$L_- = L^{(\zeta_1)}, \;\; L_+ = L^{(\zeta_2)}$

Dirac monopole configuration near $(0,0,0)$:

$$\boxed{\phi(z,\bar{z},y)|_{z,y\to 0} \sim \frac{\gamma}{(|z|^2+y^2)^{\frac{1}{2}}}}, \quad \exp(2\pi\imath\gamma) = \zeta_1^{-1}\zeta_2$$

$\gamma = (\gamma_1, \ldots, \gamma_l) \in \mathfrak{h}$ - charges of l Dirac monopoles.

Hecke transformation $\Xi(z)$:

$$L_+(z) = \Xi(z) L_-(z) \Xi^{-1}(z)$$

$$\mathfrak{g} = \mathrm{sl}(2,\mathbb{C}) \text{ HT} : (s=0) \to (s=1), \quad \nabla^{(0)} \to \nabla^{(1)}$$

$$\Xi(u,z) \sim \begin{pmatrix} \theta_0(z-2u;2\tau) & \theta_0(z+2u;2\tau) \\ \theta_1(z-2u;2\tau) & \theta_1(z+2u;2\tau) \end{pmatrix}$$

$$\nabla^{(1)} = Ad_{\Xi}(\nabla^{(0)})$$

$\theta_j(z;\tau) = (-1)^j \sum_{k\in\mathbb{Z}} \exp\Big(2\pi i((k+\frac{j}{2})^2\tau + (k+\frac{j}{2})z\Big)$, $(j=0,1)$

$\Xi(z)$ -twist - transformation of the IRF (dynamical) r-matrix to the Vertex r-matrix.