Supersymmetric Mechanics with Spin Variables and Nahm Equations

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Outline

Outline

Outline

Outline

Outline

Outline

Preface

Warm-up: the (1,4,3) + (4,4,0) model

The (1,4,3) + (3,4,1) model

Examples

Special multi-center case

Summary and outlook

Preface

Supersymmetric Quantum Mechanics (SQM) (Witten, 1981) is the simplest ($d = 1$) supersymmetric theory:

- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Extended SUSY in $d = 1$ is specific: dualities between various supermultiplets (Gates Jr. & Rana, 1995, Pashnev & Toppan, 2001) nonlinear "cousins" of off-shell linear multiplets (E.I., S.Krivonos, O.Lechtenfeld, 2003, 2004), etc.
- The models of superconformal mechanics are relevant to AdS_2/CFT_1, standing for CFT_1, and to supersymmetric *black holes*, accounting for their near-horizon geometry.

- The standard approach to setting up SQM models:
  1. Start from a few irreps of $d = 1$ supersymmetry;
  2. Construct their invariant Lagrangian (with the second- and first-order kinetic terms for bosonic and fermionic $d = 1$ fields);
  3. Quantize and define the relevant Hamiltonian and supercharges;
  4. Find the relevant (at least double-degenerate) spectrum and wave functions.

- **Example**: the original Witten's SQM is $\mathcal{N} = 2$ SQM associated with the supermultiplets (**1, 2, 1**), the numerals counting physical **bosonic**, physical **fermionic** and auxiliary **bosonic** $d = 1$ fields.

- Recently, a new kind of SQM models with $\mathcal{N} = 4, d = 1$ supersymmetry was discovered and studied (Fedoruk, E.I., Lechtenfeld, 2009, 2010, 2011). They involve two coupled irreducible $\mathcal{N}=4$ multiplets, one **dynamical** (standard) and one "**semi-dynamical**", with the first-order $d=1$ Wess-Zumino term for the bosonic variables.

- Upon quantization, the semi-dynamical variables play the role of spin degrees of freedom parametrizing a fuzzy manifold. In the simplest case they are $\mathrm{SU}(2)$ doublets, and one obtains the standard fuzzy sphere (Madore, 1992). Hence the alternative name "**spin multiplet**" for the semi-dynamical multiplet.

- Why $\mathcal{N}=4$ and not, e.g., $\mathcal{N}=2$? Just because $\mathcal{N}=4$ SUSY possesses non-abelian $\mathrm{SU}(2)$ symmetry: spin variables are in fact a sort of target $\mathrm{SU}(2)$ harmonic variables.

- The first examples of these $\mathcal{N}=4$ SQM models were constructed as a one-particle limit of a new type of $\mathcal{N}=4$ super Calogero models. They describe an off-shell coupling of a **dynamical** (**1,4,3**) multiplet to a **gauged** (**4,4,0**) **spin** multiplet. The latter finally carry only two independent bosonic variables due to gauge freedom and some algebraic constraint.

- They inherit the superconformal $D(2,1;\alpha)$ invariance of the parent super Calogero models and realize a novel mechanism of generating a conformal potential $\sim x^{-2}$ for the dynamical bosonic variable, with a quantized strength.

- This construction was generalized by replacing the dynamical (**1,4,3**) multiplet with a (**4,4,0**) or a (**3,4,1**) one, still keeping the (**4,4,0**) spin multiplet (E.I., Konyushikhin, Smilga, 2010; Bellucci, Krivonos, Lechtenfeld, Sutulin, 2010).

- The larger number of dynamical bosons allowed for Lorentz-force-type couplings to **non-abelian** self-dual background gauge fields in a manifestly $\mathcal{N}=4$ supersymmetric fashion. The presence of the spin variables proved to be **crucial** for going beyond abelian backgrounds.

- What about making use of other $\mathcal{N}=4$ multiplets to represent the target spin degrees of freedom? Recently, the multiplet with the off-shell contents (**3,4,1**) was used as the spin one (Fedoruk, E.I., Lechtenfeld, 1204.4474), still with the (**1,4,3**) multiplet as dynamical.

- A new striking feature: in this case $\mathcal{N}=4$ supersymmetry amounts to a Nahm-like equation for the $\mathrm{SU}(2)$ triplet of the bosonic spin variables, with the physical bosonic variable of the dynamical multiplet playing the role of "evolution" parameter.

- Actually, this triplet is restricted by some algebraic constraint, just leaving us with **two** independent bosonic variables parametrizing the "spin space", just as in the case of the (**4,4,0**) spin multiplet. These two spin multiplets are related by a type of quantum Hopf mapping.

- Both sorts of the spin multiplets have a natural off-shell superfield description in Harmonic $\mathcal{N}=4, d=1$ superspace (E.I., Lechtenfeld, 2003) which is $d=1$ version of Harmonic $\mathcal{N}=2, d=4$ superspace (Galperin, E.I., Kalitzin, Ogievetsky, Sokatchev, 1984).

Warm-up: the $(1,4,3) \oplus (4,4,0)$ model

The off-shell superfield action is a sum of three parts

$$S = S_{\mathcal{X}} + S_{FI} + S_{WZ}, \tag{1}$$

$$S_{\mathcal{X}} = -\tfrac{1}{2}\int dt\, d^4\theta\, \mathcal{X}^2\,, \; S_{WZ} = -\tfrac{1}{2}\int \mu_A^{(-2)}\, \mathcal{V}\, \tilde{\mathcal{Z}}^+\, \mathcal{Z}^+\,, \; S_{FI} = -\tfrac{i}{2}\, c\int \mu_A^{(-2)}\, V^{++}\,.$$

$\mathcal{N} = 4$ superfield $\mathcal{X}$ describes the off-shell multiplet **(1, 4, 3)**

$$D^i D_i\, \mathcal{X} = 0\,, \qquad \bar{D}_i \bar{D}^i\, \mathcal{X} = 0\,, \qquad [D^i, \bar{D}_i]\, \mathcal{X} = 0\,.$$

Superfields $\mathcal{Z}^+, \tilde{\mathcal{Z}}^+$ are defined on the **analytic** subspace $(t_A, \theta^+, \bar{\theta}^+, u_i^{\pm})$ of the harmonic $\mathcal{N} = 4, d = 1$ superspace $(\theta^i, \bar{\theta}_i, u_i^{\pm})$. They obey the harmonic constraints

$$(D^{++} + i\, V^{++})\, \mathcal{Z}^+ = (D^{++} - i\, V^{++})\, \tilde{\mathcal{Z}}^+ = 0$$

and describe a gauge-covariantized version of the $\mathcal{N}=4$ multiplet (**4,4,0**). The gauge analytic superfields V^{++} and $\mathcal{V}$ are subject to the gauge freedom

$$\mathcal{V}' = \mathcal{V} + D^{++}\lambda^{--}\,, \quad V^{++\prime} = V^{++} - D^{++}\lambda, \quad \mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^+\,,$$

where λ, λ^{--} are arbitrary analytic superfield parameters. The superfield $\mathcal{V}$ is an analytic "prepotential" for the (**1, 4, 3**) multiplet, $\mathcal{X} = \int du\, \mathcal{V}$, and $\mu^{(-2)}$ is the measure of integration over the analytic superspace.

After passing to WZ gauge $V^{++} = 2i\,\theta^+\bar\theta^+ A(t_A)$, integrating over θ and eliminating auxiliary fields from both (**1, 4, 3**) and (**1, 4, 3**) multiplets, as well as some rescalings of the involved fields, the action becomes

$$S = \int dt \left[p\,\dot x + i\left(\bar\psi_k\dot\psi^k - \dot{\bar\psi}_k\psi^k\right) + \tfrac{i}{2}\left(\bar z_k\dot z^k - \dot{\bar z}_k z^k\right) - H\right]$$

$$H = \tfrac{1}{4}\,p^2 + \frac{1}{4}\,\frac{(\bar z_k z^k)^2}{4x^2} + \frac{\psi^i\bar\psi^k z_{(i}\bar z_{k)}}{x^2}\,.$$

The “gauge” field $A(t)$ is the Lagrange multiplier for the first-class constraint

$$D^0 - c \equiv \bar z_k z^k - c \approx 0\,, \qquad (2)$$

which should be imposed on the wave functions in quantum case. We rewrite the potential term as

$$\frac{(\bar z_k z^k)^2}{4x^2} = \frac{(y_a y_a)}{x^2}\,, \quad y_a = \tfrac{1}{2}\,\bar z_i(\sigma_a)^i{}_j z^j\,, \quad y^2 - c^2/4 \approx 0\,.$$

The mapping $(z_i, \bar z^i) \to y^a$ is none other than Hopf $S^3 \to S^2$ fibration. Upon quantization, $z_i \to \hat z^i = \partial/\partial\bar z^i$, the triplet $\hat y^a$ becomes SU(2) generators

$$[\hat y_a, \hat y_b] = i\,\epsilon_{abc}\hat y_c\,,$$

while the S^2 sphere condition $y^2 - c^2/4 \approx 0$ becomes the Casimir condition

$$\hat y_a\hat y_a = \tfrac{1}{2}\,\hat{\bar z}_k\hat z^k\left(\tfrac{1}{2}\,\hat{\bar z}_k\hat z^k + 1\right) \Rightarrow \tfrac{1}{2}\,c\left(\tfrac{1}{2}\,c + 1\right)$$

Thus after quantization one is left with the “fuzzy” sphere in the target space, c being “fuzzyness” (or SU(2) spin for the SU(2) irrep wave functions).

The wave functions satisfy the constraint

$$D^0\Phi = \hat{\bar z}_i \hat z^i \Phi = \bar z_i \frac{\partial}{\partial \hat z^i}\,\Phi = c\,\Phi \;\;\rightarrow\;\; \Phi(x,\psi,\hat z_i) = \phi_{k_1\ldots k_c}(x)\,\hat z^{k_1}\ldots \hat z^{k_c}\,.$$

Thus in our case the wave function carries an irreducible spin $c/2$ representation of the group SU(2), being an SU(2) spinor of the rank c. In contradistinction, in most models of the ordinary (super)conformal mechanics, these w.f. are singlets of the internal symmetry group.

The quantum supercharges have the simple form

$$Q^i = \hat p\hat\psi^i - i\,\frac{\hat z^{(i}\hat{\bar z}^{k)}\hat\psi_k}{\hat x}\,, \qquad \bar Q_i = \hat p\hat{\bar\psi}_i + i\,\frac{\hat z_{(i}\hat{\bar z}_{k)}\hat{\bar\psi}^k}{\hat x}\,.$$

$$\{Q^i, \bar Q_k\} = 2\delta^i_k H\,, \qquad H = \frac{1}{4}\left(\hat p^{\,2} + \frac{\hat g}{\hat x^2}\right),$$

$$\hat g \equiv \frac{1}{2}D^0\left(\frac{1}{2}D^0 + 1\right) + 4\hat z^{(i}\hat{\bar z}^{k)}\hat\psi_{(i}\hat{\bar\psi}_{k)}\,.$$

Taking into account that $D^0 = c$ on wave functions, we observe that the "semi-dynamical" spin variables enter the Hamiltonian and supercharges only through the composite SU(2) triplet $\hat z^{(i}\hat{\bar z}^{k)} \sim \hat y_a$. Is it possible to find a formulation in which this variable is elementary and appears from scratch? The answer is YES (Fedoruk, E.I., Lechtenfeld, 1204.4474 [hep-th]).

The (1,4,3) ⊕ (3,4,1) model

The superfield action is

$$S = -\frac{1}{2}\int \mu_H\, \mathcal{X}^2 + \frac{i}{2}\int \mu_A^{(-2)}\, \mathcal{V}\,(L^{++} + c^{++}) - \frac{i}{2}\int \mu_A^{(-2)}\, \mathcal{L}^{(+2)}(L^{++}, u)\,.$$

Here, $c^{++} = c^{ik} u_i^+ u_k^+$. The constrained $\mathcal{N} = 4$ superfield $\mathcal{X}$ describes the multiplet (**1, 4, 3**), the analytic gauge $\mathcal{V}$ superfield is the (**1, 4, 3**) prepotential and the analytic constrained superfield L^{++}, $D^{++}L^{++} = 0$, describes the multiplet (**3, 4, 1**) $\propto (v^{(ik)}, \psi^k, \bar{\psi}^k, B)$. After eliminating all auxiliary fields except B, the bosonic part of the component action takes the form

$$S_{bos} = \int dt\left[\dot{x}\dot{x} - \frac{1}{4}(v_a + c_a)(v_a + c_a) - \mathcal{A}_a \dot{v}_a + B\,(x - \mathcal{U})\right],$$

$$\mathcal{U}(v) := \int du \frac{\partial \mathcal{L}^{++}}{\partial v^{++}}, \quad \mathcal{A}_a(v) := \int du\,(u^+\sigma_a u^-)\,\frac{\partial \mathcal{L}^{++}}{\partial v^{++}},$$

$$\Delta\,\mathcal{U} = \Delta\,\mathcal{A}_b = 0\,,\ \partial_a \mathcal{A}_a = 0\,,\ \mathcal{F}_{ab} := \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a = -\epsilon_{abc}\partial_c \mathcal{U}\,,$$

$$\mathcal{U} = \mathcal{U}_n := g_0 + \sum_{s=1}^{n} \frac{g_s}{|\vec{v} - \vec{k}_s|} \quad \Longrightarrow \quad \vec{\mathcal{A}} = \sum_{s=1}^{n} \vec{\mathcal{A}}_s\,,$$

$$\vec{\mathcal{A}}_s = g_s \frac{\vec{n}_s \times (\vec{v} - \vec{k}_s)}{|\vec{v} - \vec{k}_s|\left(|\vec{n}_s||\vec{v} - \vec{k}_s| + \vec{n}_s(\vec{v} - \vec{k}_s)\right)}\,, \quad \text{multi} - \text{monopoles on } \mathbb{R}^3\,.$$

Quantization: Hamiltonian constraints

The relevant Hamiltonian constraints and the Hamiltonian are

$$\pi_a \equiv p_a + \mathcal{A}_a \approx 0\,, \;\; h \equiv x - \mathcal{U} \approx 0\,,$$
$$H = \tfrac{1}{4}\,p^2 + \tfrac{1}{4}\,(v_a + c_a)(v_a + c_a) + \lambda_a \pi_a + Bh\,,$$

where λ_a and B are the Lagrange multipliers. Poisson brackets of these constraints:

$$[\pi_a, \pi_b]_P = -\mathcal{F}_{ab}\,, \qquad [\pi_a, h]_P = \partial_a \mathcal{U}\,, \tag{3}$$

Determinant of the matrix of the r.h.s. of (3) is $(\partial_a \mathcal{U} \partial_a \mathcal{U})^2 \neq 0$, implying that all four constraints are second class. The Dirac brackets are

$$[x, p]_D = 1\,, \quad [v_a, x]_D = 0\,,$$
$$[v_a, p]_D = \frac{\partial_a \mathcal{U}}{\partial_p \mathcal{U} \partial_p \mathcal{U}}\,, \quad [v_a, v_b]_D = -\epsilon_{abc}\,\frac{\partial_c \mathcal{U}}{\partial_p \mathcal{U} \partial_p \mathcal{U}}\,.$$

We end up with two independent physical phase variables (x and p) and two independent spin variables, hidden in v_a. The constraint $x - \mathcal{U}(v) \approx 0$ can be treated as the equation defining a two-dimensional surface in the $\mathbb{R}^3$ manifold parametrized by the variables v_a.

Quantization: Nahm equations

The Dirac brackets $[v_a, p]_D$ and $[v_a, v_b]_D$ amount to the equations

$$[p, v_a]_D = \frac{1}{2}\,\epsilon_{abc}\,[v_b, v_c]_D \quad \Rightarrow \quad v_a' = -\frac{1}{2}\,\epsilon_{abc}\,[v_b, v_c]_D$$

for $v_a = v_a(x, \ell_b)$, such that $[\ell_a, x]_D = [\ell_a, p]_D = 0$.

These are none other than the generalized (the so called "SDiff(Σ_2)") Nahm equations (see, e.g., Ward, 1990; Dunajski, 2003).

It turns out that these Nahm equations and their quantum counterpart just guarantee the **very existence** of the $\mathcal{N}=4$ supersymmetry in models with the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ spin multiplet, both at the classical and the quantum levels.

Taking into account the hamiltonian constraints including $x - \mathcal{U}(v) \approx 0$, the classical supercharges and Hamiltonian are calculated to be

$$Q^i = p\,\chi^i + i\,(v_a + c_a)\,\sigma_a^{ik}\chi_k\,, \qquad \bar{Q}_i = p\,\bar{\chi}_i - i\left(v_a + c_a\right)\sigma_{a\,ik}\bar{\chi}^k\,,$$

$$H = \tfrac{1}{4}\,p^2 + \tfrac{1}{4}\,(v^a + c^a)\left(v_a + c_a\right) - \chi_i\sigma_a^{ik}\bar{\chi}_k\,\partial_a\mathcal{U}/(\partial_p\mathcal{U}\partial_p\mathcal{U})\,.$$

With the Dirac brackets for the bosonic phase variables as above and with $\{\chi^i, \bar{\chi}_k\}_D = -\frac{i}{2}\,\delta^i_k$, these operators form the classical $\mathcal{N} = 4$ superalgebra

$$\{Q^i, \bar{Q}_k\}_D = -2i\,\delta^i_k H\,, \qquad \{Q^i, Q^k\}_D = [Q^i, H]_D = 0\,.$$

Direct calculation shows that these relations are in fact valid just because of the Nahm equations.

In the classical case these equations are just the consequence of the underlying (Poisson -Dirac) structure. But what about quantum case? The Dirac brackets among the variables p and v_a are in general highly nonlinear and it is not obvious how to quantize them. It turns out that requiring the validity of the basic $\mathcal{N} = 4$ superalgebra relations in the quantum case is again equivalent to the proper quantum version of the Nahm equations.

The quantum expressions for the supercharges are uniquely found to be

$$\hat{Q}^i = \hat{p}\,\hat{\chi}^i + i\,(\hat{v}_a + c_a)\,\sigma_a^{ik}\hat{\chi}_k\,, \qquad \hat{\bar{Q}}_i = \hat{p}\,\hat{\bar{\chi}}_i - i\left(\hat{v}_a + c_a\right)\sigma_{a\,ik}\hat{\bar{\chi}}^k\,,$$

$$\{\hat{\chi}^i, \hat{\bar{\chi}}_k\} = \tfrac{1}{2}\hbar\delta^i_k\,.$$

One calculates their anticommutators and finds, e.g.,

$$\{\hat{Q}^i, \hat{Q}^j\} \quad = \quad i\,\sigma_a^{ij}\left([\hat{p}, \hat{v}_a] - \tfrac{1}{2}\,\epsilon_{abc}\,[\hat{v}_b, \hat{v}_c]\right)\hat{\chi}^n\hat{\chi}_n\,.$$

It is vanishing only provided the quantum version of the Nahm equation holds

$$[\hat{p}, \hat{v}_a] = \tfrac{1}{2}\,\epsilon_{abc}[\hat{v}_b, \hat{v}_c] \quad \Rightarrow \quad \hbar\frac{\partial}{\partial\hat{x}}\,\hat{v}_a = \tfrac{i}{2}\,\epsilon_{abc}[\hat{v}_b, \hat{v}_c]\,.$$

The same equation arises from requiring the $\{Q, \bar{Q}\}$ anticommutator to contain only SU(2) singlet part $\sim H_q$. The relevant quantum Hamiltonian is uniquely determined:

$$\hat{H} = \tfrac{1}{4}\,\hat{p}^2 + \tfrac{1}{4}\,(\hat{v}_a + c_a)\left(\hat{v}_a + c_a\right) - i\hbar^{-1}[\hat{p}, \hat{v}_a]\hat{\chi}_i\sigma_a^{ik}\hat{\bar{\chi}}_k\,.$$

Thus, quite similarly to the classical case, after quantization the quantum operators $\hat{v}_a$ must be subjected to the operator Nahm equations.

Examples: one-monopole case

$$\mathcal{U}_1 := \frac{g}{|\vec{v}-\vec{k}|}\,,\ \mathcal{A}_a = g\,\frac{\epsilon_{abc}n_b(v_c-k_c)}{|\vec{v}-\vec{k}|\left(|\vec{n}||\vec{v}-\vec{k}|+\vec{n}(\vec{v}-\vec{k})\right)},\ (\vec{n}=\vec{k}/|\vec{k}|\,,\vec{k}=-\vec{c})\,.$$

Constraint:

$$x = \frac{g}{|\vec{v}-\vec{k}|} \quad \Rightarrow \quad \ell_a\ell_a = g^2\,, \quad \text{for} \quad \ell_a = x\,(v_a-k_a) = g\frac{v_a-k_a}{|\vec{v}-\vec{k}|}\,.$$

The new phase variables x, p, ℓ_a satisfy

$$[x,p]_D = 1\,,\ [\ell_a,x]_D = 0\,,\ [\ell_a,p]_D = 0\,,\ [\ell_a,\ell_b]_D = \epsilon_{abc}\ell_c\,.$$

Thus the variables ℓ_a parametrize a sphere S^2 with the radius g and generate SU(2) group with respect to the Dirac brackets. The Nahm equations are evidently satisfied by $v_a = \frac{\ell_a}{x} + k_a$. After quantization:

$$\ell_a \quad \rightarrow \quad \hat{\ell}_a\,, \quad [\hat{\ell}_a,\hat{\ell}_b] = i\hbar\,\epsilon_{abc}\hat{\ell}_c\,, \quad \hat{\ell}_a\hat{\ell}_a = \hbar^2 n(n+1), \qquad (4)$$

hence $\hat{\ell}_a$ are $(2n+1)\times(2n+1)$ matrices. As a result, the wave function has $(2n+1)$ components and describes a non–relativistic spin n conformal particle. The quantum Nahm equation becomes the standard matrix SU(2) Nahm equation (still with x as the evolution parameter).

Since the Nahm equations are satisfied at the classical and quantum levels, the $\mathcal{N}=4$ superalgebra relations always hold. The quantum supercharges and the Hamiltonian are:

$$Q^i = p\,\chi^i + i\,\frac{\hat{\ell}_a\,\sigma_a^{ik}\chi_k}{x}\,, \qquad \bar{Q}_i = p\,\bar{\chi}_i - i\,\frac{\hat{\ell}_a\,\sigma_{a\,ik}\bar{\chi}^k}{x}$$

$$H = \frac{1}{4}\left(p^2 + \frac{\hat{\ell}_a\hat{\ell}_a}{x^2}\right) + \frac{\hat{\ell}_a\,\chi_i\sigma_a^{ik}\bar{\chi}_k}{x^2}\,.$$

The wave function is:

$$\Psi^A(x,\chi^i) = \phi^A(x) + \chi^i\,\psi_i^A(x) + \chi^i\chi_i\,\varphi^A(x)\,,$$

Here $A = 1,\ldots 2n$ is an index of the irreducible SU(2) representation with $\hat{\ell}_a$ as generators. W.r.t. the full $\mathrm{SU}(2)$ generated by $J_a = \hat{\ell}_a - \chi_i\sigma_a^{ik}\chi_k$, the bosonic wave functions $\phi^A(x)$ and $\varphi^A(x)$ form two spin n SU(2) irreps, while the fermionic functions $\psi_i^A(x)$ carry two SU(2) irreps, with SU(2) spins $n \pm \frac{1}{2}$.

This system exhibits an extended $\mathcal{N}=4$ superconformal symmetry OSp(4|2) and so supplies an example of $\mathcal{N}=4$ superconformal mechanics.

Examples: two-monopole cases

$$\mathcal{U}_2 := \frac{g_1}{|\vec{v} - \vec{k}_1|} + \frac{g_2}{|\vec{v} - \vec{k}_2|}\,, \quad \vec{k}_1 = (0,0,k_1)\,,\ \vec{k}_2 = (0,0,k_2)\,.$$

The idea is to pass to the new spin variables, such that the spinning sector in the phase space is separated from the dynamical "space" sector (x, p). We pass to the new variables as

$$\ell_3 := \frac{g_1(v_3 - k_1)}{|\vec{v} - \vec{k}_1|} + \frac{g_2(v_3 - k_2)}{|\vec{v} - \vec{k}_2|}\,, \quad \varphi := \arctan\left(\frac{v_2}{v_1}\right).$$

They commute with the variables of the dynamical sector:

$$[\ell_3, p]_D = [\ell_3, x]_D = 0\,, \quad [\varphi, p]_D = [\varphi, x]_D = 0\,.$$

The variables ℓ_3 and φ are conjugate to each other:

$$[\varphi, \ell_3]_D = 1\,.$$

The variable ℓ_3 in the bosonic limit commutes with the Hamiltonian and so generates U(1) symmetry. Both SU(2) and OSp(4|2) are now broken, only $\mathcal{N} = 4, d = 1$ Poincaré supersymmetry and U(1) R-symmetry survive.

One should still express v_a as $v_a = v_a(x, \varphi, \ell_3)$, since supercharges and Hamiltonian involve just v_a. Even in the classical case these inverse relations for the two-center case can be found only as **a series** in ℓ_3:

$$v_\pm := v_1 \pm i v_2 = V(x, \ell_3) e^{\pm i\varphi} , \qquad v_3 = W(x, \ell_3) . \tag{5}$$

No problem with the validity of the classical Nahm equations in this case. What about the quantum case?

The basic step in passing to the quantum supercharges from the classical ones is to perform the Weyl-ordering of the latter (Smilga, 1987). In our case this prescription amounts to replacing

$$v_\pm \Rightarrow \hat{v}_\pm = \langle V(\hat{x}, \hat{\ell}_3)\, e^{\pm i\hat{\varphi}} \rangle_W , \qquad v_3 \Rightarrow \hat{v}_3 = W(\hat{x}, \hat{\ell}_3) . \tag{6}$$

and making use of the Moyal bracket, when calculating the (anti)commutators between supercharges and Hamiltonian.

Rather surprisingly, in the two-center model it proves **insufficient** just to Weyl-order the classical expressions to obtain the correct quantum $\mathcal{N} = 4$ superalgebra. One can explicitly check that the quantum Nahm equations which guarantee the validity of $\mathcal{N} = 4$ superalgebra **are not satisfied** with (6). Only when v_a are **linear** in ℓ_a, the quantum Nahm equations are satisfied. But this is possible only in the one-center and some special multi-center cases.

The way out is as follows. We assume that the above functions $V(x, \ell_3)$, $W(x, \ell_3)$ are just $\hbar = 0$ approximation of the correct quantum functions which contain correction terms of the higher order in $\hbar$

$$\begin{aligned} V(x,\ell_3) &\quad \to \quad \tilde{V}(x,\ell_3,\hbar) = V + \hbar V_1 + \hbar^2 V_2 + \ldots, \\ W(x,\ell_3) &\quad \to \quad \tilde{W}(x,\ell_3,\hbar) = W + \hbar W_1 + \hbar^2 W_2 + \ldots . \end{aligned}$$

Then we require that the Nahm equations (with the Dirac brackets being replaced by the Moyal ones) are still satisfied with these modified Weyl symbols $\tilde{v}_\pm$ and $\tilde{v}_3$:

$$[p, \tilde{v}_a]_M = -\frac{\partial \tilde{v}_a}{\partial x} = \tfrac{1}{2}\, \epsilon_{abc}[\tilde{v}_b, \tilde{v}_c]_M \,. \tag{7}$$

Thus we propose to correct the quantum operators in higher orders in the expansion in $\hbar$, in such a way that the full operator Nahm equations are satisfied, while the limit $\hbar \to 0$ still yields the classical system.

In this setting, the Moyal-Nahm equations (7) for the modified Weyl symbols of $\hat{v}_\pm, \hat{v}_3$ amount to the equations for the coefficient functions $V_n(x, \ell_3)$ and $W_n(x, \ell_3)$. Solving these equations, we can find the complete solutions for the quantum operators. This recursion procedure is self-consistent and yields the correct quantum supercharges and Hamiltonian as power series in $\hbar$.

Special multi-center case

There exists a potential with few centers for which the original spin variables v_a are linear in ℓ_a in both the classical and the quantum cases. It reads:

$$\tilde{\mathfrak{U}} = \frac{g}{k}\,\mathrm{arcoth}\left(\frac{|\vec{v}+\vec{k}|+|\vec{v}-\vec{k}|}{2k}\right), \qquad \vec{k}=(0,0,k).$$

This potential satisfies the Laplace equation $\Delta\tilde{\mathfrak{U}}=0$. Besides the two poles at $\vec{v}=\pm\vec{k}$, it possesses the third pole at $\vec{v}=0$.

We split v_a into the "radial variable" x and the spin ones ℓ_a as

$$v_1 = f_1(x)\,\ell_1\,,\;\; v_2 = f_2(x)\,\ell_2\,,\;\; v_3 = f_3(x)\,\ell_3\,,$$
$$f_1 = f_2 = \frac{k}{g\,\sinh(kx/g)}\,,\;\; f_3 = \frac{k}{g}\coth(kx/g)\,.$$

The Dirac brackets of p, x and v_a induce the following ones for p, x, ℓ_a

$$[x,p]_{_D} = 1\,, \qquad [\ell_a, x]_{_D} = 0\,, \qquad [\ell_a, p]_{_D} = 0\,, \qquad [\ell_a, \ell_b]_{_D} = \epsilon_{abc}\ell_c\,,$$

whereas the constraint $x - \tilde{\mathfrak{U}} \approx 0$ becomes the 2-sphere condition

$$\ell_a\ell_a = g^2\,.$$

The passing to the quantum case is straightforward:

$$\ell_a \Rightarrow \hat{\ell}_a\,, \quad [\hat{\ell}_a, \hat{\ell}_b] = i\hbar\, \epsilon_{abc}\hat{\ell}_c\,, \qquad g^2 \Rightarrow \hbar^2\, n(n+1)\,, \quad 2n \in \mathbb{N}\,.$$

The quantum Nahm equations for $\hat{v}_a(\hat{x}, \hat{\ell}_b)$ are satisfied as a consequence of the fact that the functions f_1, f_2 and f_3 satisfy the Euler equations

$$f_1' = -f_2 f_3\,, \qquad f_2' = -f_1 f_3\,, \qquad f_3' = -f_1 f_2\,.$$

The explicit form of the quantum supercharges and Hamiltonian is as follows

$$\begin{aligned} \hat{Q}^i &= \hat{p}\,\hat{\chi}^i + \tfrac{ik}{g}\sinh^{-1}(\tfrac{k\hat{x}}{g})\Big[\hat{\ell}_1\,\sigma_1^{ik}\hat{\chi}_k + \hat{\ell}_2\,\sigma_2^{ik}\hat{\chi}_k \\ &\qquad + \Big(\cosh(\tfrac{k\hat{x}}{g})\,\hat{\ell}_3 + \tfrac{cg}{k}\sinh(\tfrac{k\hat{x}}{g})\Big)\,\sigma_3^{ik}\hat{\chi}_k\Big]\,, \\ \hat{\bar{Q}}_i &= \hat{p}\,\hat{\bar{\chi}}_i - \tfrac{ik}{g}\sinh^{-1}(\tfrac{k\hat{x}}{g})\Big[\hat{\ell}_1\,\sigma_{1\,ik}\hat{\bar{\chi}}^k + \hat{\ell}_2\,\sigma_{2\,ik}\hat{\bar{\chi}}^k \\ &\qquad + \Big(\cosh(\tfrac{k\hat{x}}{g})\,\hat{\ell}_3 + \tfrac{cg}{k}\sinh(\tfrac{k\hat{x}}{g})\Big)\,\sigma_{3\,ik}\hat{\bar{\chi}}^k\Big]\,, \end{aligned}$$

$$\begin{aligned} \hat{H} &= \tfrac{1}{4}\,\hat{p}^2 + \tfrac{k^2}{4g^2}\sinh^{-2}(\tfrac{k\hat{x}}{g})\Big[(\hat{\ell}_1)^2 + (\hat{\ell}_2)^2 + \Big(\cosh(\tfrac{k\hat{x}}{g})\,\hat{\ell}_3 + \tfrac{cg}{k}\sinh(\tfrac{k\hat{x}}{g})\Big)^2\Big] \\ &\quad + \tfrac{k^2}{g^2}\sinh^{-2}(\tfrac{k\hat{x}}{g})\Big[\cosh^2(\tfrac{k\hat{x}}{g})\Big(\hat{\ell}_1\hat{\chi}_i\sigma_1^{ik}\hat{\bar{\chi}}_k + \hat{\ell}_2\hat{\chi}_i\sigma_2^{ik}\hat{\bar{\chi}}_k\Big) + \hat{\ell}_3\hat{\chi}_i\sigma_3^{ik}\hat{\bar{\chi}}_k\Big] \end{aligned}$$

In the limit $k \to 0$, the one-monopole $\mathcal{N} = 4$ model is reproduced.

Summary and outlook

- We presented new versions of $\mathcal{N}=4$ mechanics, which couple a dynamical ("coordinate") (**1,4,3**) multiplet to a semi-dynamical ("spin") (**3,4,1**) multiplet. An on-shell constraint involving a harmonic potential on $\mathbb{R}^3$ leaves only two independent bosonic fields in the spin multiplet. They parametrize some two-dimensional fuzzy surface in $\mathbb{R}^3$.
- For the one-center potential and the special multi-center potential, the spin variables generate an SU(2) algebra and parametrize the fuzzy two-sphere. These quantum models are formulated in a closed form, while the one with a general two-center potential is given in terms of a power series expansion in $\hbar$.
- Most remarkable feature is the occurrence of the Nahm equations for the three-vector spin variable as a consequence of the Dirac brackets of the constraints, with the bosonic field of the dynamical multiplet playing the role of the evolution parameter.
- We discovered a strict correspondence between these Nahm equations and the presence of $\mathcal{N}=4$ supersymmetry in the model, classically and quantum mechanically. **The Nahm equations guarantee extended supersymmetry**.

Some further problems to be explored:

- It would be interesting to study the general multi-center solution of the Laplace equation $\Delta\mathcal{U} = 0$ for the basic potential $\mathcal{U}(v)$. For this case one may expect the spin variables to parametrize some fuzzy Riemann surface and form a nonlinear deformed algebra.

- In our particular models the supersymmetry generators are **linear** in the fermionic variables. In the more general case of $\mathcal{N}=4$ supersymmetry generators **cubic** in the fermions the Nahm equations might get supplemented by additional relations to ensure full extended supersymmetry.

- Finally, it remains to investigate other combinations of dynamical and semi-dynamical $\mathcal{N}=4$ multiplets for describing spin variables, utilizing for instance the **nonlinear** (**3,4,1**) multiplet (E.I., Lechtenfeld, 2003; E.I., Krivonos, Lechtenfeld, 2004).

- Relations to branes, black holes, AdS/CFT, integrable structures in $\mathcal{N} = 4$ SYM?

THANK YOU FOR YOUR KIND ATTENTION!