Parent BRST approach to higher spin gauge fields

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Based on:

M.G. arXiv:1204.1793, arXiv:1012.1903

G. Barnich, M.G., arXiv:1009.0190, arXiv:0905.0547

K. Alkalaev, M.G., arXiv:1105.6111

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Appropriate Language for Higher spin gauge theories?

Metric-like approach and its BRST extension –

Rather natural and simple *Fronsdal, 1979*

String-inspired BRST approach. *Ouvry, Stern, 1986, Bengtsson, 1986, Henneaux, Teitelboim, 1986,*

More recent contributions: *Pashnev, Buchbinder, Sagnotti, Tsulaia, Francia, Bekaert...*

Frame-like "unfolded" approach –

Naturally appeares at the nonlinear level Makes symmetries manifest.

Allows for powerful homological technique (e.g. so-called σ_--cohomology).

Mainly developed by *Vasiliev, 1988,...*

More recent contributions: *Sezgin, Sundell, Alkalaev, Skvortsov, Boulanger, ...*

Main point – metric like BRST and unfolded approaches are actually unified if one carefully applies Batalin–Vilkovisky formalism and local BRST cohomology technique...

Moreover, the exchange of methods and ideas turns out to be quite fruitful!

Batalin-Vilkovisky formalism:

Given equations T_a, gauge symmetries R^i_α, reducibility relations,.... the BRST differential:

$$s = \delta + \gamma + \ldots\,, \qquad s^2 = 0\,, \quad \mathrm{gh}(s) = 1$$

$$\delta = T_a \frac{\partial}{\partial \mathcal{P}_a} + Z^a_A \mathcal{P}_a \frac{\partial}{\partial \pi^A} \ldots\,, \qquad \gamma = c^\alpha R^i_\alpha \frac{\partial}{\partial \phi_i} + \ldots\,.$$

δ – (Koszule-Tate) restriction to the stationary surface

γ – implements gauge invariance condition

ϕ^i – fields, c^α – ghosts,

$\mathcal{P}_a$ – ghost momenta, π^A – reducibility ghost momenta

$$\mathrm{gh}(\phi^i) = 0\,, \quad \mathrm{gh}(c^\alpha) = 1\,, \quad \mathrm{gh}(\mathcal{P}_a) = -1\,, \quad \ldots$$

BRST differential completely defines the theory.

Equations of motion and gauge symmetries can be read off from s:

$$s\mathcal{P}_a|_{\mathcal{P}_a=0,\, c^\alpha=0\,,\ldots} = 0\,, \quad \delta_\epsilon \phi^i = (s\phi^i)|_{c^\alpha=\epsilon^\alpha,\, \mathcal{P}_a=0,\, \ldots}$$

If the theory is Lagrangian then: $T_i = \frac{\delta S_0}{\delta \phi^i}$, reducibility relations $R^i_\alpha T_i = 0$

so that $Z^i_\alpha = R^i_\alpha$

Natural bracket structure (antibracket)

$$\left(\phi^i, \mathcal{P}_j\right) = \delta^i_j \qquad \left(c^\alpha, \mathcal{P}_\beta\right) = \delta^\alpha_\beta$$

BV master action

$$s = \left(\cdot, S_{BV}\right), \qquad S_{BV} = S_0 + \mathcal{P}_i R^i_\alpha c^\alpha + \dots$$

Master equation:

$$\left(S_{BV}, S_{BV}\right) = 0 \quad \Longleftrightarrow \quad s^2 = 0$$

Example: YM theory

Fields: A_μ, C (with values in the Lie algebra)

Antifields: $A^{*\mu}$, C^*

Gauge part BRST differential: $\gamma A_\mu = \partial_\mu C + [A_\mu, C]$

Master action:

$$S_{BV} = S_0 + \int d^n x \,\mathrm{Tr}[A^{*\mu}(\partial_\mu C + [A_\mu, C]) + \frac{1}{2} C^* [C, C]]$$

In the context of local gauge field theory:

Jet space: coordinates Ψ^A – fileds, ghosts, antifields

$$x^\mu, \ \xi^\mu, \ \Psi^A, \ \Psi^A_\mu, \ \Psi^A_{\mu\nu}, \ \ldots \qquad \xi^\mu \equiv dx^\mu$$

Total derivative:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \Psi^A_\mu \frac{\partial}{\partial \Psi^A} + \Psi^A_{\mu\nu} \frac{\partial}{\partial \Psi^A_\nu} + \ldots$$

BRST differential is an evolutionary vector field:

$$[\partial_\mu, s] = 0\,, \qquad s\Psi^A = s^A[\Psi, x]$$

Local functionals:

Quotient space: $\qquad f[\Psi] \sim f[\Psi] + \partial_\mu j^\mu[\Psi]$

In a local field theory – local BRST cohomology encode physically interesting quantities such as conserved currents/global symmetries, anomalies, consistent deformations etc.

Local BRST cohomology in local functionals, local forms, evolutionary (poly)vector fields etc.

Although jet-space BV is extremely useful it can be quite restrictive:
– Boundary dynamics (e.g. AdS/CFT, asymptotic symmetries)
– Coordinate-free formulation (e.g. for gravity)
– Manifest realization of symmetries

An alternative: *Vasiliev, 1988, . . . , 2005*

Unfolded formalism

Fields: differential forms Φ^a

Equations of motion: $d\Phi^a = Q^a(\Phi)$, $Q^a(\Phi)$ – wedge product function.

Consistency: $Q^2 = 0$ where $Q = Q^a(\Phi)\frac{\partial}{\partial \Phi^a}$

Free Differential Algebras, *Sullivan 1977, d'Auria, Fre, 1982. . .*

Advantages:

– manifestly coordinate free

– first order

– useful in analyzing global symmetries

– inevitable for nonlinear higher spin theories *Vasiliev, 1989,. . . , 2003*

Open issues:

1) No systematic procedure to “unfold” a given theory

2) In spite of various algebraic similarities the relation between jet space BV and unfolded approaches remains unclear

3) Known unfolded forms for sufficiently general higher spin fields are quite involved

4) Even for Lagrangian systems constructing unfolded Lagrangians is rather an art than a systematic procedure

For linear theories 1),2) were mainly resolved within the first quantized BRST approach *Barnich, M.G., Semikhatov, Tipunin, 2004, Barnich, M.G. 2006*.
In particular, BRST extension of unfolded systems *Barnich, M.G. 2005*

3) Mixed-symmetry HS Minkowski *Alkalaev, M.G. Tipunin 2008*
on AdS space *Alkalaev, M.G. 2009, 2011*
Particular conformal fields *Bekaert, M.G. 2009*

AKSZ sigma models

Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994

Ingredients:

M - supermanifold (target space) with coordinates Ψ^A:

Ghost degre – gh()

(odd) Poisson bracket – $\{\cdot\,,\,\cdot\}$, $\mathrm{gh}(\{\cdot\,,\,\cdot\}) = -n+1$

"BRST potential" $S_M(\Psi)$, $\mathrm{gh}(S_M) = n$, master equation $\{S_M, S_M\} = 0$

(QP structure: $Q = \{\cdot\,, S_M\}$ and $P = \{\cdot\,,\,\cdot\}$)

$\mathcal{X}$ - supermanifold (source space)

Ghost degree gh()

$\boldsymbol{d}$ – odd vector field, $\boldsymbol{d}^2 = 0$, $\mathrm{gh}(\boldsymbol{d}) = 1$

Tipically, $\mathcal{X} = T[1]X$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$, $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\mu = 0, \ldots n-1$

BV master action

$$S_{BV} = \int d^n x d^n \theta \left[\chi_A(\Psi(x,\theta)) \boldsymbol{d}\Psi^A(x,\theta) + S_M(\Psi(x,\theta))\right]$$

$\chi_A(\Psi)$ – symplectic potential

BV antibracket

$$\big(F, G\big) = \int d^n x d^n \theta \left(\frac{\delta^R F}{\delta \Psi^A(x,\theta)} E^{AB} \frac{\delta G}{\delta \Psi^B(x,\theta)}\right).$$

$E^{AB} = \left\{\Psi^A, \Psi^B\right\}$ – Poisson bivector

Master equation:

$$\big(S_{BV}, S_{BV}\big) = 0\,, \qquad \mathrm{gh}(S_{BV}) = 0$$

BRST differential:

$$s^{AKSZ}\Psi^A(x,\theta) = \boldsymbol{d}\Psi^A(x,\theta) + Q^A(\Psi(x,\theta))\,, \qquad Q^A = \left\{\Psi^A, S_M\right\}$$

Dynamical fields, those of vanishing ghost degree

$$\Psi^A(x,\theta) = \overset{0}{\Psi}{}^A(x) + \overset{1}{\Psi}{}^A_\mu(x)\theta^\mu + \ldots \qquad \mathrm{gh}(\overset{k}{\Psi}{}^A_{\mu_1\ldots\mu_k}) = \mathrm{gh}(\Psi^A) - k$$

If $\mathrm{gh}(\Psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\Psi}{}^A_{\mu_1\ldots\mu_k}(x)$ is dynamical.

If $\mathrm{gh}(\Psi^A) \geqslant 0 \ \forall \ \Psi^A$ then BV-BRST extended FDA.

Otherwise BV-BRST extended FDA with constraints.

Equations of motion (nonlagrangian) AKSZ:

$$\{,\}\,,\, S_M \quad \rightarrow \quad \text{nilpotent} \;\; Q = Q^A \frac{\partial}{\partial \Psi^A}\,.$$

No relation between gh(Q) and dim X ! (Recall gh(S_M) $= n =$ dim X)

BV-BRST extension of unfolded form with constraints

Examples:

Chern-Simons: *Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994*

Target space M:

$M = \mathfrak{g}[1]$, $\mathfrak{g}$ – Lie algebra with invariant inner product.

e_i –basis in $\mathfrak{g}$, C^i – coordinates on $\mathfrak{g}$, $\mathrm{gh}(C^i) = 1$, $C = C^i e_i$

$$S_M = \frac{1}{6}\langle C, [C, C]\rangle\,, \qquad \left\{C^i, C^j\right\} = \langle e_i, e_j\rangle^{-1}$$

Source space:

$\mathcal{X} = T[1]X$, X – 3-dim manifold. Fied content

$$C^i(x,\theta) = \lambda^i(x) + \theta^\mu A^i_\mu(x) + \theta^\mu\theta^\nu A^{*i}_{\mu\nu} + \theta^\mu\theta^\nu\theta^\rho\lambda^{*i}_{\mu\nu\rho}$$

BV action

$$S_{BV} = \int d^3x d^3\theta\left(\frac{1}{2}\langle C, \boldsymbol{d}C\rangle + \frac{1}{6}\langle C, [C, C]\rangle\right) = \int \frac{1}{2}\langle A, \boldsymbol{d}A\rangle + \frac{1}{6}\langle A, [A, A]\rangle) + \ldots$$

1d AKSZ systems

Target space M:

BFV extended phase space, $\{,\}$ –Poisson bracket, $S_M = \Omega$ – BRST charge, $\{\Omega, \Omega\} = 0$ – BFV master equation, in addition: function H, $\{H, \Omega\} = 0$ – BRST invariant Hamiltonian

Source space $\mathcal{X} = T[1](\mathbb{R}^1)$, coordinates t, θ

BV action *M.G., Damgaard, 1999*

$$S_{BV} = \int dt d\theta (\chi_A \boldsymbol{d} \psi^A + \Omega - \theta H)$$

Integration over θ gives BV for the Hamiltoninan action *Fisch, Henneaux, 1989, Batalin, Fradkin 1988*.

Example: coordinates on M: $\tilde{c}, \tilde{\mathcal{P}}, \tilde{x}^\mu, \tilde{p}_\mu$, BRST charge $\Omega = \tilde{c}(\tilde{p}^2 - m^2)$,

$$S_{BV} = \int dt d\theta (\tilde{p}_\mu \boldsymbol{d} \tilde{x}^\mu + \tilde{\mathcal{P}} \boldsymbol{d} \tilde{c} + \tilde{c}(p^2 - m^2)) = \int dt (p_\mu \dot{x}^\mu + \lambda(p^2 + m^2)) + \dots$$

$\tilde{c}(t, \theta) = c(t) + \theta \lambda(t)$, $\tilde{x}^\mu(t, \theta) = x^\mu(t) + \theta p^\mu_*(t)$, ...

AKSZ is neither Lagrangian nor Hamiltonian

AKSZ model: $M, S_M, \{,\}$ and $T[1]X, \boldsymbol{d}$.

Let $X = X_S \times \mathbb{R}^1$

$$\Omega_{BFV} = \int d^{n-1}x d^{n-1}\theta \left[\chi_A(\Psi(x,\theta))\boldsymbol{d}\Psi^A(x,\theta) + S_M(\Psi(x,\theta))\right]$$

$$\{\,\cdot\,,\,\cdot\,\}_{BFV} = \int d^{n-1}x d^{n-1}\theta \,\{\,\cdot\,,\,\cdot\,\} \qquad \{\Omega_{BFV}, \Omega_{BFV}\}_{BFV} = 0\,.$$

Barnich, M.G, 2003

Higher BRST charges

$\chi(\Psi)\boldsymbol{d}\Psi + S_M(\Psi(x,\theta))$ – integrand of S_{BV} as inhomogeneous form on X,

$X_k \subset X$ – dimension-k submanifold

$$\Omega_{X_k} = \int_{X_k} L_{AKSZ} = \int_{X_k} (\chi\boldsymbol{d}\Psi + S_M)$$

In particular, $\Omega_{BFV} = \Omega_{X_S}$, $S_{BV} = \Omega_X$

– At the level of equations of motion one induces AKSZ sigma model on any $X_k \subset X$. Useful for "replacing space-time". E.g.

Generalized superspace *Vasiliev 2002*

Natural way to relate AdS, Ambient, and Conformal picture *Barnich M.G. 2006, Bekaert M.G. 2009*

AdS/CFT for interacting HS fields *Vasileiv, 2012*

– Locally in X and M *Barnich, M.G. 2009*

$$H^g(s^{AKSZ}, \text{local functionals}) \cong H^{g+n}(Q, C^\infty(M))$$

Isomorphism sends $f \in C^\infty(M)$ to functional $F = \int f$.

Compatible with the bracket.

– If M finite dimensional and $n > 1$ – the model is topological.

Parent formulation (Equations of motion level)

Barnich, M.G. 2010

Barnich, M.G., Semikhatov, Tipunin, 2004

Starting point theory:

Fields, ghosts, ghosts for ghosts, antifields, etc.: $\psi^I(z)$

Jet space M for BV formulation: coordinates $\Psi^A = \{z^a, \xi^a \equiv dz^a, \psi^I_{(a)}\}$
(short-hand $\psi^I_{(a)} = \{\psi^I, \psi^I_a, \psi^I_{a_1 a_2}, \dots\}$)

Horizontal differential: $d_H = \xi^a \partial_a$

BRST differential: s – vector field on M, $[d_H, s] = 0$

Basic object $\tilde{s} = -d_H + s$ *Brandt, 1997*

Parent formulation

AKSZ sigma model:

– target space M equipped with $\tilde{s} = -d_H + s$

– source space x^μ, θ^μ.

BRST differential

$$s^P \boldsymbol{\Psi}^A(x,\theta) = (\boldsymbol{d} + \tilde{s})\boldsymbol{\Psi}^A(x,\theta)$$

Fields:

$$\Psi^A(x,\theta) = \{\psi^I_{(a)}(x,\theta), \quad z^a(x,\theta), \quad \xi^a(x,\theta)\}$$

Dynamical fields (gh() = 0):

$$\overset{k}{\psi}{}^I_{(a)\mu_1\dots\mu_k}(x) \quad \text{gh}(\psi^I) = k \geqslant 0, \qquad z^a(x) = \overset{0}{z}{}^a(x)\,, \qquad e^a_\mu(x) = \overset{1}{\xi}{}^a_\mu(x)$$

In fact: we are dealing with parametrized version.

$z^a(x)$ – space-time coordinates understood as fields

e^a_μ – frame field components.

Gauge transf. for z^a: $\delta z^a = \xi^a$.

Gauge condition $z^a = \delta^a_\mu x^\mu$ give unparametrized version:

$$s^P \Psi^A(x,\theta) = (\boldsymbol{d} - \theta^a \partial_a + s)\Psi^A(x,\theta) \qquad \text{note:} \quad e^a_\mu = \delta^a_\mu$$

Recall: ∂_a – target space total derivative.

What is the relation between the starting point theory and its parent formulation?

Generalized auxiliary fields and equivalent reductions

At the Lagrangian level:

χ^i, χ^*_i are generalized auxiliary fields for S_{BV} if they are conjugate in the antibracket and equations $\left.\frac{\delta S_{BV}}{\delta \chi^i}\right|_{\chi^*_i=0} = 0$ can be algebraically solved for χ^i. *Dresse, Grégoire, Henneaux, 1990*

At the level of equations of motion: φ^α, v^a, w^a

$$(sw^a)|_{w^a=0} = 0 \quad \Leftrightarrow \quad v^a = V^a[\varphi]$$

v^a, w^a – *generalized auxiliary fields*. *Barnich, M.G., Semikhatov, Tipunin, 2004*

Reduced system:

$$s_R\phi^\alpha = s\phi^\alpha|_{w=0,v=V[\phi]}\,, \qquad (s_R)^2 = 0$$

Can be seen as reduction to the surface:

$$w^a = 0, \qquad v^a - V^a[\varphi] = 0\,,$$

Equivalence = Elimination of generalized auxiliary fields

(Local) BRST cohomology is invariant.

E.g. observables, global symmetries, consistent interactions, anomalies, possible Lagrangians, are isomorphic.

Parent formulation is equivalent to the starting point one. All the fields $\overset{0}{\psi}{}^I_{(a)\mu_1..\mu_k}(x)$ save for $\overset{0}{\psi}{}^I(x)$ are generalized auxiliary.

Simple algebraic reason: using extra variables y^a pack fields as

$$\tilde{\psi}^I(y,\theta) = \sum_{m,k} \overset{k}{\psi}{}^I_{b_1\dots b_m|a_1\dots a_k} y^{b_1}\dots y^{b_m}\theta^{a_1}\dots\theta^{a_k}$$

Poincare lemma says: exists basis $1, f_i, g_i$ in $[y^a, \theta^a]$ such that

$$\theta^a \frac{\partial}{\partial y^a} f_i = g_i$$

. If $\tilde{\psi}^I = \overset{0}{\psi}{}^I + F^i f_i + G^i g_i$ fields F^i, G^i are generalized auxiliary.

Reduction to unfolded formulation

BRST differential decomposition: $s = \delta + \gamma + \ldots$, where δ implements equations of motion. For a regular theory new coordinates on M

$$\phi^\lambda, \quad T^i, \quad \mathcal{P}^i$$

such that ϕ^λ are coordinates on the stationary surface and $\delta\mathcal{P}^i = T^i$.

Fields $T^i(x,\theta), \mathcal{P}^i(x,\theta)$ are generalized auxiliary for parent formulation

$$s^P_{on-shell}\phi^\lambda(x,\theta) = (\boldsymbol{d} + \tilde{\gamma})\phi^\lambda(x,\theta)\,, \qquad \tilde{\gamma} = \gamma_{\text{on}-\text{shell}} - d_H$$

As gh$(\phi^\lambda) \geqslant 0$ the equations of motion and gauge symmetries are that of some FDA.

General prescription to unfold a given gauge theory. However:

1) Not a standard Vasiliev unfolded form but usually a nonminimal one.

2) Parametrized version

Parent Lagrangians

Starting point theory:

Fields, ghosts, ghosts for ghosts (but no antifields!): $\psi^I(y)$

Gauge part of BRST differential: γ (assumption $\gamma^2 = 0$)

Lagrangian: $L[\psi, y]$, $\gamma L = \partial_\mu j^\mu[\psi, y]$.

Parent Lagrangian

Jet space N with coordinates $\Psi^\alpha = \{\psi^I_{(a)}, z^a, \xi^a\}$.

Equipped with: ghost degree, $d_H = \xi^a \partial_a$, $\tilde{\gamma} = -d_H + \gamma$

Lagrangian potential $\widehat{L}(\psi, z, \xi)$:

$$\widehat{L} = L_n + L_{n-1} + \ldots + L_0\,, \qquad \text{where} \quad L_n = \xi^{n-1} \ldots \xi^0 L[\psi, z]$$

$L_{n-1}, \dots, L_0$ through "Descent equation" $(-d_H + \gamma)\widehat{L} = 0$:

$$\gamma L_n = d_H L_{n-1}$$

$$\gamma L_{n-1} = d_H L_{n-2}$$

$$\dots = \dots$$

$$\gamma L_0 = 0$$

$\widehat{L}$ represents Lagrangian as a $\tilde{\gamma} = -d_H + \gamma$ cohomology class.

Introduce antifields $\Lambda_\alpha = \{\Lambda_I^{(a)}, \pi_a, \rho_a\}$ and the canonical (anti)bracket:

$$\mathrm{gh}(\Lambda_I^{(a)}) = n - 1 - \mathrm{gh}(\psi^I_{(a)})\,, \qquad \mathrm{gh}(\pi_a) = n - 1\,, \qquad \mathrm{gh}(\rho_a) = n - 2$$

$$\left\{\psi^I_{(a)}, \Lambda_J^{(b)}\right\} = \delta^I_J \delta^b_a\,, \qquad \{y^a, \pi_b\} = \delta^a_b\,, \qquad \{\xi^a, \rho_b\} = \delta^a_b$$

Supermanifold $M = T^*[n-1]N$.

Lagrangian parent formulation

Target space: $M = T^*[n-1]N$, canonical degree $1-n$ bracket, BRST potential

$$S_M = \Lambda_\alpha \widetilde{\gamma} \Psi^\alpha + \widehat{L}(\Psi)$$

BV master action:

$$S_{BV} = \int d^n x d^n \theta [\Lambda_\alpha (\boldsymbol{d} + \widetilde{\gamma}) \Psi^\alpha + \widehat{L}(\phi)]$$

S_{BV} satisfies master equation $\left(S_{BV}, S_{BV}\right) = 0\,.$

$\Lambda_\alpha(x,\theta)$ – sources for parent BRST transformation. Unify momenta, Lagrange multipliers, BV antifields.

Fronsdal fields

Fields and Ghosts (gauge parameters):

$$\phi^{a_1 \dots a_s}\,, \qquad C^{a_1 \dots a_{s-1}}$$

$$\phi = \frac{1}{s!} p_{a_1} \dots p_{a_s} \phi^{a_1 \dots a_s}\,, \qquad C = \frac{1}{(s-1)!} p_{a_1} \dots p_{a_{s-1}} C^{a_1 \dots a_{s-1}}\,.$$

$$TT\phi = 0\,, \quad TC = 0\,, \qquad T \equiv \frac{\partial}{\partial p_a} \frac{\partial}{\partial p^a}$$

Gauge part of the BRST differential

$$\gamma\phi = p^a \partial_a C$$

Target space coordinates $z^a, \xi^a \equiv dz^a, \phi(p,y), C(p,y)$

$$\phi(p,y) = \phi(p) + \phi_a(p)y^a + \frac{1}{2}\phi_{ab}(p)y^a y^b + \ldots\,,$$
$$C(p,y) = C(p) + C_a(p)y^a + \frac{1}{2}C_{ab}(p)y^a y^b + \ldots\,.$$

On-shell version ϕ, C-totally traceless:

$$SC = \Box C = S\phi = \Box\phi = 0\,, \qquad S = \frac{\partial}{\partial y^a}\frac{\partial}{\partial p_a}\,, \quad \Box = \frac{\partial}{\partial y^a}\frac{\partial}{\partial y^a}\,.$$

Dynamical fields and ghosts

$$C(x,\theta|y,p) = \lambda(x|y,p) + \theta^a A_a(x|y,p) + \ldots\,,$$
$$\phi(x,\theta|y,p) = F(x|y,p) + \theta^a \overset{1}{\phi}(x|y,p) + \ldots\,,$$

Equations and gauge symmetries: *Barnich, M.G., Semikhatov, Tipunin, 2004*

$$(\boldsymbol{d} - \sigma)A = 0 \qquad (\boldsymbol{d} - \sigma)F + S^\dagger A = 0\,,$$

$$\delta A = (\boldsymbol{d} - \sigma)\lambda, \qquad \delta F = S^\dagger \lambda\,,$$

$\sigma = \xi^a \frac{\partial}{\partial y^a}$ $S^\dagger = p^a \frac{\partial}{\partial y^a}$

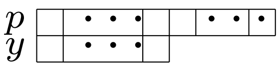
Cohomological results

Barnich, M.G., Semikhatov, Tipunin, 2004

All variables are contractible pairs for γ save for z^a, ξ^a and

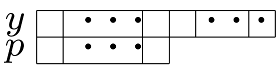
Generalized connections $\mathcal{C}(y,P) \subset C(y,p)$

$$S^\dagger \mathcal{C} = 0\,,$$



Generalzed curvatures (de Witt - Freedman) $R(y,p) \subset \phi(y,p)$

$$y^a \frac{\partial}{\partial p^a} R = 0\,,$$



$\tilde{\gamma} = d_H + \gamma$ reduces to

$$\tilde{\gamma}_{\text{red}} \mathcal{C} = \sigma \mathcal{C} + \Pi \sigma \bar{\sigma} R\,, \qquad \tilde{\gamma}_{\text{red}} R = \Pi \sigma R\,,$$

HS Russian formula

HS version of the familiar YM

Stora, 1983,.....

$$YM \qquad \tilde{\gamma}_{\text{red}} \tilde{C} = \frac{1}{2}[\tilde{C}, \tilde{C}] + F\,, \qquad \tilde{\gamma}_{\text{red}} F = \ldots$$

$$GR \qquad \tilde{\gamma}_{\text{red}} \xi^a = \xi^a_c \xi^c\,, \qquad \gamma_{\text{red}} \xi^a_b = \xi^a_c \xi^c_b - \frac{1}{2} \xi^c \xi^d R^a_{b\,cd}\,,$$

$$\tilde{\gamma}_{\text{red}} R = \ldots$$

Reduced system determied by $s_{\text{red}} = \boldsymbol{d} + \widetilde{\gamma}_{\text{red}}$ *(Lopatin, Vasiliev 1988)*:

(unfolded form)

$$(\boldsymbol{d} - \Pi\sigma)\widehat{F} = 0, \qquad (\boldsymbol{d} - \sigma)\widehat{A} = -\sigma\bar{\sigma}\,\Pi\widehat{F}$$

where

$$\mathcal{C}(x,\theta|y,p) = \lambda(x|y,p) + \theta^a \widehat{A}_a(x|y,p) + \ldots,$$

$$R(x,\theta|y,p) = \widehat{F}(x|y,p) + \theta^a \ldots + \ldots,$$

Frame-like Lagrangian

Fields and ghosts (gauge parameters):

$$\phi^{a_1\dots a_s}\,, \qquad C^{a_1\dots a_{s-1}}$$

$$\phi = \frac{1}{s!}p_{a_1}\dots p_{a_s}\phi^{a_1\dots a_s}\,, \quad C = \frac{1}{(s-1)!}p_{a_1}\dots p_{a_{s-1}}C^{a_1\dots a_{s-1}}\,.$$

$$TT\phi = 0\,, \quad TC = 0\,, \qquad T \equiv \frac{\partial}{\partial p_a}\frac{\partial}{\partial p^a}$$

Gauge part of the BRST differential

$$\gamma\phi = p^a\partial_a C$$

Fronsdal Larangian: *Fronsdal, 1979*

$$L = \frac{1}{2}\langle\partial_a\phi, \partial^a\phi\rangle - \frac{1}{2}\langle\bar{p}^a\partial_a\phi, \bar{p}^b\partial_b\phi\rangle +$$
$$+ \langle p_a\partial^a D, \bar{p}^b\partial_b\phi\rangle - \langle\partial_a D, \partial^a D\rangle - \frac{1}{2}\langle\bar{p}^a\partial_a D, \bar{p}^b\partial_b D\rangle,$$

$\bar{p}^a = \frac{\partial}{\partial p_a}$ and $D = T\phi$.

Lagrangian Parent formulation:

Target space supermanifold: $\phi(y,p)$, $\mathcal{C}(y,p)$, z^a, ξ^a

Simplification: eliminate contractible pairs for γ such that

$$T\phi(y,p) = 0\,, \quad SC(y,p) = 0\,, \qquad\qquad S = \frac{\partial}{\partial p_a}\frac{\partial}{\partial y^a}$$

Target space version of the "traceless gauge"

Alvarez, Blas, Garriga, Verdaguer (2006), Skvortsov, Vasiliev (2007)

Lagrangian: *Skvortsov, Vasiliev (2007)*

$$L = \frac{1}{2}\langle \partial_a \phi, \partial^a \phi \rangle|_{y=0} - \frac{1}{2}\langle S\phi, S\phi \rangle|_{y=0}$$

The Lagrangian potential $(-d_H + \gamma)\widehat{L} = 0$

$$\widehat{L} = \mathcal{V}L + \mathcal{V}_a J^a + \frac{1}{2}\mathcal{V}_{ab} J^{ab}$$

$$\mathcal{V}_{a_1 \ldots a_k} = \frac{1}{(n-k)!}\epsilon_{a_{a_1} \ldots a_k b_1 \ldots b_{n-k}} \xi^{b_1} \ldots \xi^{b_{n-k}}$$

Possible solution

$$J^a = \langle \phi, p^a \Box C \rangle|_{y=0} - \langle \phi, \partial^a S^\dagger C \rangle|_{y=0}$$

$$J^{ba} = \frac{1}{2}\left[\langle p^b C, p^a \Box C - \partial^a S^\dagger C \rangle|_{y=0} - \langle S^\dagger C, p^b \partial^a C \rangle|_{y=0} - (a \leftrightarrow b)\right],$$

All ingredients for the parent Lagrangian:

Supermanifold ϕ, C, z^a, ξ^a, $\widetilde{\gamma} = -d_H + \gamma$, Lagrange potential $\widehat{L}$

Equivalence: $\widehat{L} \to L + \widetilde{\gamma} K$.

Cohomological results *Bekaert, Boulanger, 2005*:

All variables are γ-contractible pairs save for z^a, ξ^a

HS connections $\mathcal{C}(y,p) \subset C(y,p)$, Off-shell HS curvatures $\widehat{R} \subset \phi(y,p)$,

Fronsdal tensors $\mathcal{F} \subset \phi(y,p)$.

$$\mathcal{F} = \text{independent components of} \quad (\Box\phi - S^\dagger S + S^\dagger S^\dagger T)\phi(y,p)$$

Better choice for $\widehat{L}$ $(s \geqslant 2)$:

$$\widehat{L} = \frac{1}{2}\mathcal{V}_{ab}\left[\langle C_a, C_b\rangle - \langle p^a C_d, p^b C_d\rangle\right] + M$$

where M vanishes when trivial pairs for $\tilde{\gamma}$ are eliminated.

Finally

$$\widehat{L}_{red} = \frac{1}{2}\mathcal{V}_{ab}\left[\langle \mathcal{C}_a, \mathcal{C}_b\rangle - \langle p^a \mathcal{C}_d, p^b \mathcal{C}_d\rangle\right]$$

Elimination results in

$$S_R[e, \omega, \Lambda] = \int \langle \Lambda, \boldsymbol{d}e - \sigma\omega\rangle + \widehat{L}_{red}(\omega)\,,$$

$$C(x, \theta|p) = \overset{0}{C}(x|p) + \theta^b e_b(x, a) + \theta^b\theta^d \ldots + \ldots$$
$$C_a(x, \theta|p) = \overset{0}{C}_a(x|p) + \theta^b \omega_{a|p}(x, a) + \theta^b\theta^d \ldots + \ldots$$

In fact ω is auxiliary as well. Paramerizing $n-2$ form Λ in terms of 1 form $\widehat{\omega}$ one gets:

Vasiliev, 1980

$$S_{frame}[e, \widehat{\omega}] = \int d^n x \,\langle \widehat{\omega}, y^a \frac{\partial}{\partial x^a} e - \frac{1}{2}\widehat{\omega}\rangle' = \int d^n\theta d^n x\, \mathcal{V}_{cab}\langle \frac{\partial}{\partial p^c}\widehat{\omega}_a, \frac{\partial}{\partial p^b}(\boldsymbol{d}e - \frac{1}{2}\sigma\widehat{\omega})\rangle$$

Off-shell nonlinear system

Recall (parent EOM's): *Barnich, M.G., Semikhatov, Tipunin, 2004*

$$(\boldsymbol{d}-\sigma)A=0 \qquad (\boldsymbol{d}-\sigma)F+S^{\dagger}A=0\,, \quad A,F \text{ – totally traceless}$$

$\sigma=\xi^a\frac{\partial}{\partial y^a},\ S^{\dagger}=p^a\frac{\partial}{\partial y^a}$

Off-shell version (A,F – unconstrained) is a linearization of:

$$dA+\frac{1}{2}[A,A]_*=0\,, \qquad dF+[A,F]_*=0\,,$$

Vasiliev, 2005

around a particular solution $A_0=\theta^b p_b,\ F_0=\frac{1}{2}\eta^{ab}p_a p_b$.

Here

$$[G,H]_*=G*H-(-1)^{|G||H|}H*G$$

– Weyl $*$-commutator determined by $[y^a,p_b]_*=\delta^a_b$.

Perfect example (especially the spin-2 sector) to illustrate

A-"frame", F-"metric" – synthetic frame-like-metric-like formulation!

Can be seen as a BFV master equation $\Omega * \Omega = 0$ fo

$$\Omega = \boldsymbol{d} + \theta^{\mu} A(x|y,p) + cF(x|y,p)$$

Parent form of the quantized scalar particle propagating in HS background.

Can be related to the Fedosov quantization *Fedosov, 1994* of the particle model.

In this way one also implements double-tracelessness condition *M.G., 2006*

Analogous first-quantized description of the nonlinear off-shell formulation for conformal fields *Segal, 2002*

Off-shell constraints and gauge symmetries for AdS HS

Formal version of the ambient space+oscilators:

Coordinates Y^A, P_A

Functions: formal series in Y and polynomials in P

Weyl star product

$$Y^A * P_B - P_B * Y^A = \delta^A_B$$

$sp(2)$ Lie algebra with the basis e_i, $[e_i, e_j] = U^k_{ij}$

$$[e_2, e_1] = 2e_1\,, \quad [e_2, e_3] = -2e_3\,, \quad [e_1, e_3] = e_2$$

Equations of motion for $A = dx^\mu A_\mu(x|Y, P)$, $F_i = F_i(x|Y, P)$

$$\boldsymbol{d}A + \frac{1}{2}[A, A]_* = 0\,, \quad \boldsymbol{d}F_i + [A, F_i]_* = 0\,, \qquad [F_i, F_j]_* - U^k_{ij} F_k = 0$$

Gauge symmetries

$$\delta_\lambda F_i = [F_i, \lambda]_*\,, \qquad \delta_\lambda A = \boldsymbol{d}\lambda + [A, \lambda]_*$$

Linearize around background solution:

$$A^0_\mu = \theta^\mu \omega^B_{\mu A}(x)(Y^A + V^A)P_B\,,$$

$$F^0_1 = \frac{1}{2}P\cdot P\,, \qquad F^0_2 = (Y+V)\cdot P\,, \qquad F^0_3 = -\frac{1}{2}(Y+V)\cdot(Y+V)$$

$\omega^B_{\mu A}(x)$ – flat AdS connection, V^A – compensator

"Twisted" version of the familiar $sp(2)$- representation

AdS geometry through embedding:

$$X = \{X \subset \mathbb{R}^{n+1} : \eta_{AB}X^A X^B + 1 = 0\}$$

On $\mathbb{R}^{n+1}$ flat $o(n-1,2)$ metric η, metric connection ∇_0, "tautological" vector field $V_0 = X^A \frac{\partial}{\partial X^A}$.

By pulling back $T\mathbb{R}^{n+1}$ to X one gets: fiberwise metric, flat $o(n-1,2)$ connection ∇, fixed section such that V

$$d\omega + \omega\omega = 0\,, \quad \eta_{AB}V^A V^B + 1 = 0\,.$$

Frame $e^A_\mu = \partial_\mu V^A + \omega^A_{\mu B}V^B$.

Represent linearized system as $\Omega\Psi = 0$ and $\delta\Psi = \Omega\lambda$ where

$$\Omega = \boldsymbol{d} + [A^0, \cdot]_* + \nu^i [F_i^0, \cdot]_* - \frac{1}{2}\nu^i \nu^j U_{ij}^k \frac{\partial}{\partial \nu^k}\,,$$

Barnich, M.G., 2006

"states" – $\Psi(x, \theta | Y, P, \nu)$. ν^i – ghosts.

$$\boldsymbol{d} + [A^0, \cdot]_* = \boldsymbol{d} + \theta^\mu \omega_{\mu A}^B \left(P_B \frac{\partial}{\partial P_A} - (Y^A + V^A) \frac{\partial}{\partial Y^B} \right)$$

– covariant derivative

$$\nu^i [F_i^0, \cdot]_* + \text{ghosts} =$$

$$= -\nu^1 P \cdot \frac{\partial}{\partial Y} + \nu^2 (P \cdot \frac{\partial}{\partial P} - (Y + V) \cdot \frac{\partial}{\partial Y}) - \nu^3 (Y + V) \cdot \frac{\partial}{\partial P} + \text{ghosts}$$

– BRST operator of $sp(2)$ represented on Y, P variables

If A, F_i totally traceless. $\Omega\Psi = 0$ and $\delta\Psi = \Omega\lambda$ is equivalent to Fronsdal

In fact we are dealing with AKSZ sigma-model

Target space M: generating function Ψ, $\mathrm{gh}(\Psi) = |\Psi| = 1$ for coordinates

$$\Psi(\nu, Y, P) = C(Y,P) + \nu^i F_i(Y,P) + \nu^i \nu^j G_{ij}(Y,P) + \nu^i \nu^j \nu^k G_{ijk}(Y,P)\,.$$

$$\mathrm{gh}(C) = 1, \quad \mathrm{gh}(F_i) = 0, \quad \mathrm{gh}(G_{ij}) = -1, \quad \mathrm{gh}(G_{ijk}) = -2$$

Odd vector field Q

$$Q\Psi = q\Psi + \frac{1}{2}[\Psi, \Psi]_*$$

$$q = -\frac{1}{2}\nu^i \nu^j U^k_{ij} \frac{\partial}{\partial \nu^k}\,, \qquad \text{BRST operator for } sp(2)$$

In components

$$QF_i = [F_i, C]_*\,, \qquad QC = \frac{1}{2}[C, C]_*\,,$$

$$QG_{ij} = \frac{1}{2}[F_i, F_j]_* - \frac{1}{2}U^k_{ij} F_k + [G_{ij}, C]_*\,, \qquad \ldots$$

Becuase $\mathrm{gh}(G_{ij}) < 0$ this is not FDA but FDA + constraints

Ambient picture

Parent form of the ambient space equation $(Y^A + V^A \to X^A)$

$$[F_i, F_j] = U_{ij}^k F_k\,, \quad \delta F_i = [F_i, \lambda] \qquad F_i = F_i(X,P),\ \lambda = \lambda(X,P)$$

around a particular solution $F_1^0 = \frac{1}{2} P \cdot P$, $F_2^0 = X \cdot P$, $F_3^0 = -\frac{1}{2} X \cdot X$

BFV master equation for a quntized particle on the ambient space.

This constraint system is familiar in many contexts

– Singleton on conformal boundary (quotient of the hyper-cone $X^2 = 0$)

– 2-time physics *Bars,. . .*

– Observables – HS algebra (symmetries of singleton) *Vasiliev, Eastwood*

– HS singletons and their symmetries *Bekaert, M.G., 2009*

It can be given either AdS or conformal interpretation.

– The off-shell nonlinear system is naturally defined as the AKSZ sigma-model whose target space is the direct sum of the Weyl algebra and $sp(2)$ with shifted parity.

– The formulation has well known $sp(2)$ structure realized in a manifest way.

– Can be seen as a version of the off-shell system proposed in *M.G., 2006*

– HS geometry? Connection A and multiplet of “ambient metrics” F_i?

Untouched topics

- Parent formalism can be used as a starting point to construct the theory. Used for massive and (partially) massless mixed symmetry fields on constant curvature backgrounds. Talk by *Alkalaev*
  For symmetric fields *M.G., Waldoron, 2011*

- Analogous constructions can be done for conformal fields. For bosonic singletons – explicitly done symmetries classified *Bekaert, M.G. 2009*

- Being of AKSZ form the parent Lagrangian formulation automatically gives the BFV-BRST Hamiltonian description.

- Can be naturally interpreted in terms of polymomentum DeDonder-Weyl covariant Hamiltonian formalism. Moreover, allows to systematically derive such fomulations for general gauge systems.

Conclusions

- Fruitful exchange of ideas and methods between the local BV-BRST cohomology methods, unfolded formalism, and various approaches to covariant Hamiltonian formalism.

- Provides set-up for the quantization problem along the BV quantization method. However, gauge-fixing fermion, integration measure is still to be studied, celebrated Δ-operator etc.

- Systematic way to construct unfolded formulation starting from the usual form. In particular, to generate frame-type action. Still to be done for McDowell-Mansouri-Stelle-West type HS Lagrangians.

- Generating procedure for new formulations. In particular, those that manifest one or another structure. In some sense parent formulation and its reductions make the gauge and the BRST cohomology structure manifest. For instance, gravity as a gauge theory of diffeomorphism algebra or bosonic string as a gauge theory for (regular part of) Virasoro algebra.

- As a tool to find a relevant geometry. For instance starting from metric gravity one ends up with the Cartan formulation and finds relevant curvatures just by trying to compute BRST cohomology.

- Naturally incorporates nonlinear structure, at least at the level of gauge-symmetries and off-shell constraints.