

Minimal Model Holography

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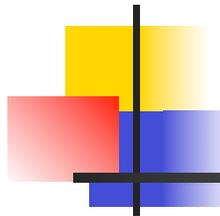
based mainly on

[MRG, R. Gopakumar](#), arXiv:1011.2986

[MRG, T. Hartman](#), arXiv:1101.2910

[MRG, R. Gopakumar, T. Hartman & S. Raju](#), arXiv:1106.1897

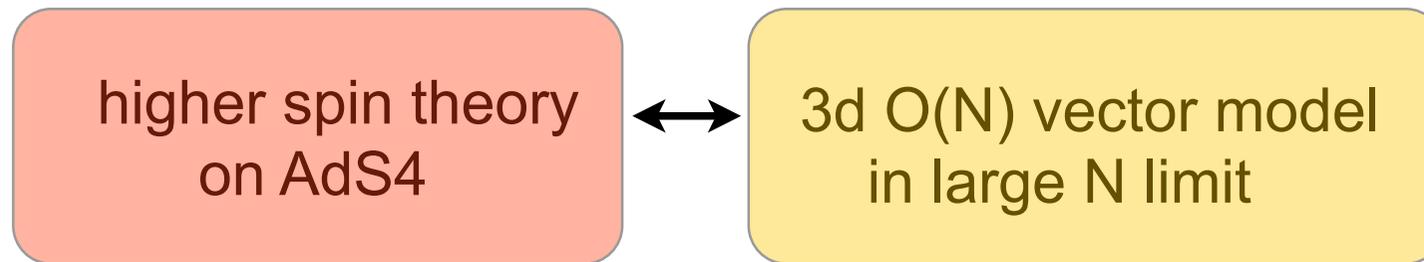
[MRG, R. Gopakumar](#), arXiv:1205.2472



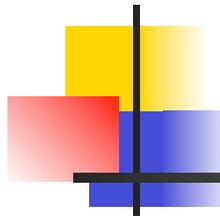
Higher spin -- CFT duality

Some years ago, a concrete proposal for a higher spin - CFT duality was made:

[Klebanov-Polyakov]
[Sezgin-Sundell]



Actually different versions, depending on whether vector model fields are **bosons or fermions** and on whether one considers **free or interacting fixed point**.



Checks of the proposal

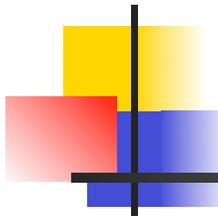
Recently **impressive checks** of the duality have been performed, in particular

3-point functions of HS fields on AdS4

have been **matched** to

3-point functions of HS currents in $O(N)$ model to leading order in $1/N$.

[Giombi & Yin]



AdS3 / CFT2

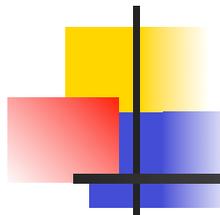
Here: describe 3d/2d CFT version of this duality.

Lower dimensional version interesting

- ▶ 2d CFTs well understood
- ▶ Higher spin theories simpler in 3d

Also, 3d conformal field theories with unbroken higher spin symmetry and finite number of d.o.f. (finite N) are necessarily **free**, but this is not the case in 2d.

[Maldacena,Zhiboedov]



3d proposal

The 3d/2d proposal takes the form

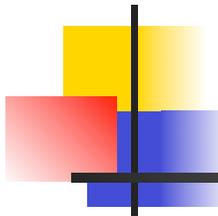
[MRG, Gopakumar]

AdS3:
higher spin theory
with a complex
scalar of mass M



2d CFT:
 $\mathcal{W}_{N,k}$ minimal models
in large N 't Hooft limit
with coupling λ

where $\lambda = \frac{N}{N+k}$ and $M^2 = -(1 - \lambda^2)$



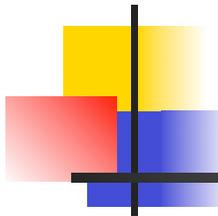
Scalars

In original version of conjecture there were **two scalars**.

Given our more detailed understanding of the symmetries (see below), it now seems that **one of the scalars** should be rather thought of as a **non-perturbative** state.

[This new point of view resolves also some puzzles regarding the structure of the correlation functions.]

[Papadodimas, Raju]
[Chang, Yin]



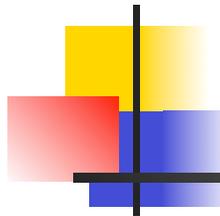
Comparison to 4d/3d case

In contrast to 4d/3d case:

1 parameter family of dual theories.

Special values:

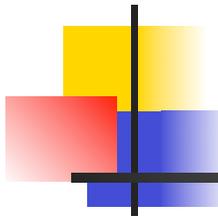
- ▶ For $\lambda = 0$ the 2d CFT is equivalent to **singlet sector of a free theory**. [MRG,Suchanek]
- ▶ For $\lambda = 1$ the resulting theory has linear **\mathcal{W}_∞ symmetry** (free bosons).



Outline

In the rest of the talk I want to explain the proposal in more detail and indicate which consistency checks have been performed.

- The HS theory in 3d
- Matching the symmetries
- The spectrum
- Conclusions



The HS theory on AdS3

The AdS3 HS theory can be described very simply.

Recall that pure gravity in AdS3: **Chern-Simons theory**
based on

$$sl(2, \mathbb{R})$$

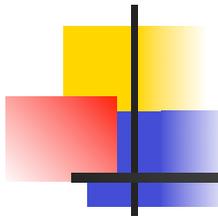
[Achucarro & Townsend]

[Witten]

Higher spin description: replace

$$sl(2, \mathbb{R}) \rightarrow \mathfrak{hs}[\lambda]$$

[Vasiliev]



Higher spin algebra

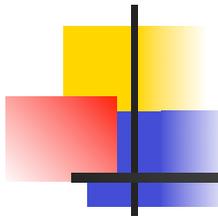
The higher spin algebra $\mathfrak{hs}[\lambda]$ is an infinite dimensional Lie algebra that can be thought of as

$$\mathfrak{hs}[\lambda] \equiv \mathfrak{sl}(\lambda, \mathbb{R})$$

[Bordemann et.al.]
[Bergshoeff et.al.]
[Pope, Romans, Shen]
[Fradkin, Linetsky]

since

$$\mathfrak{hs}[\lambda] \Big|_{\lambda=N} \cong \mathfrak{sl}(N, \mathbb{R}) \quad \text{for integer } N.$$



Asymptotic symmetries

For these higher spin theories **asymptotic symmetry algebra** can be determined following **Brown & Henneaux**, leading to **classical**

$\mathcal{W}_\infty[\lambda]$ algebra

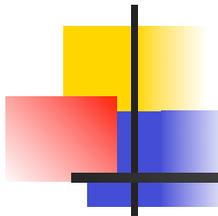
[Henneaux & Rey]
[Campoleoni et al]
[MRG, Hartman]

Extends algebra **'beyond the wedge'**:

pure gravity: $sl(2, \mathbb{R}) \rightarrow$ Virasoro

higher spin: $hs[\lambda] \rightarrow \mathcal{W}_\infty[\lambda]$

[Figueroa-O'Farrill et.al.]



Dual CFT

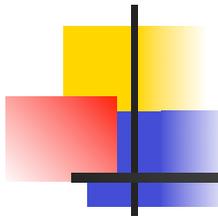
By the usual arguments, **dual CFT** should therefore have

$\mathcal{W}_\infty[\lambda]$ symmetry.

Basic idea:

$$\mathcal{W}_\infty[\lambda] = \lim_{N \rightarrow \infty} \mathcal{W}_N \quad \text{with} \quad \lambda = \frac{N}{N+k} .$$

't Hooft limit of 2d CFT!



The minimal models

The minimal model CFTs are the **cosets**

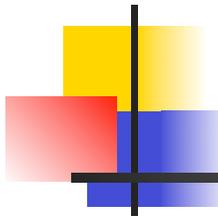
$$\mathcal{W}_{N,k} : \frac{su(N)_k \oplus su(N)_1}{su(N)_{k+1}}$$

←
e.g. Ising model (N=2, k=1)
tricritical Ising (N=2, k=2)
3-state Potts (N=3, k=1),...

with **central charge**

$$c_N(k) = (N - 1) \left[1 - \frac{N(N + 1)}{(N + k)(N + k + 1)} \right].$$

General N: **higher spin analogue** of Virasoro minimal models. [Spin fields of spin $s=2,3,\dots,N$.]



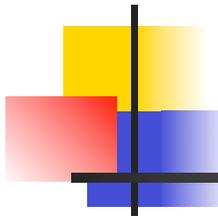
Relation of symmetries

On the face of it, the two symmetries

$$\mathcal{W}_\infty[\lambda] \quad \text{vs} \quad \lim_{N,k \rightarrow \infty} \mathcal{W}_{N,k}$$

appear to be **quite different**. However, the asymptotic symmetry analysis only determines the **classical symmetry algebra**, i.e. the algebra to leading order in $1/c$.

In order to understand above relation, we need to understand the **quantum version of this algebra**.



Quantum symmetry

The full structure of the **quantum algebra** can actually be determined completely.

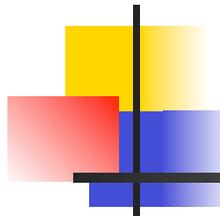
[MRG, Gopakumar]

There are **two steps** to this argument. To illustrate them consider an example. For **classical algebra**, we have

$$\begin{aligned}
 [W_m, W_n] = & 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\
 & + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}
 \end{aligned}$$

↑
↑

spin-3 field
non-linear term

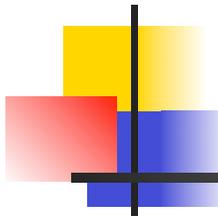


Jacobi identity

$$[W_m, W_n] = 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

spin-3 field \nearrow

\uparrow non-linear term



Jacobi identity

$$[W_m, W_n] = 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

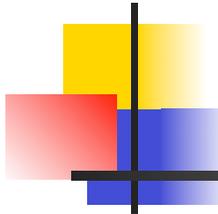
Jacobi identity determines **quantum correction**

$$[W_m, W_n] = 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ + \frac{8N_3}{c + \frac{22}{5}}(m-n)\Lambda_{m+n}^{(4)} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

where

$$\Lambda_n^{(4)} = \sum_p : L_{n-p}L_p : + \frac{1}{5}x_n L_n$$

Similar considerations apply for the other commutators.



Structure constants

The **second step** concerns structure constants. W-field can be rescaled so that

$$W \cdot W \sim \frac{c}{3} \cdot \mathbf{1} + 2 \cdot L + \frac{32}{(5c + 22)} \cdot \Lambda^{(4)} + 4 \cdot U$$

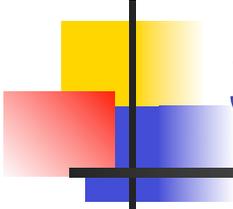
and

$$W \cdot U \sim \frac{56}{25} \frac{N_4}{N_3^2} W + \dots$$

spin-4 field

Classical analysis determines

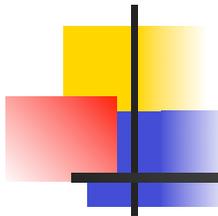
$$\frac{N_4}{N_3^2} = \frac{15 \lambda^2 - 9}{14 \lambda^2 - 4} + \mathcal{O}\left(\frac{1}{c}\right).$$



Structure constants

Classical analysis determines

$$\frac{N_4}{N_3^2} = \frac{15 \lambda^2 - 9}{14 \lambda^2 - 4} + \mathcal{O}\left(\frac{1}{c}\right).$$



Structure constants

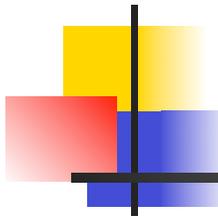
Classical analysis determines

$$\frac{N_4}{N_3^2} = \frac{15 \lambda^2 - 9}{14 \lambda^2 - 4} + \mathcal{O}\left(\frac{1}{c}\right).$$

Requirement that representation theory agrees for $\lambda = N$ with \mathcal{W}_N :

$$\frac{N_4}{N_3^2} = \frac{75 (c + 2) (\lambda - 3) (c(\lambda + 3) + 2(4\lambda + 3)(\lambda - 1))}{14 (5c + 22) (\lambda - 2) (c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}.$$

[Note: $\text{hs}[\lambda] \Big|_{\lambda=N} \cong \mathfrak{sl}(N, \mathbb{R})$ implies $\mathcal{W}_\infty[\lambda] \Big|_{\lambda=N} = \mathcal{W}_N$.]



Higher Structure Constants

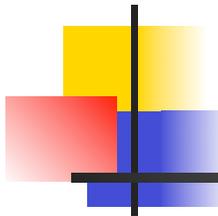
Similarly, **higher structure constants** can be determined

[Blumenhagen, et.al.] [Hornfeck]

$$C_{33}^4 C_{44}^4 = \frac{48(c^2(\lambda^2 - 19) + 3c(6\lambda^3 - 25\lambda^2 + 15) + 2(\lambda - 1)(6\lambda^2 - 41\lambda - 41))}{(\lambda - 2)(5c + 22)(c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}$$

$$(C_{34}^5)^2 = \frac{25(5c + 22)(\lambda - 4)(c(\lambda + 4) + 3(5\lambda + 4)(\lambda - 1))}{(7c + 114)(\lambda - 2)(c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}$$

$$C_{45}^5 = \frac{15}{8(\lambda - 3)(c + 2)(114 + 7c)(c(\mu + 3) + 2(4\lambda + 3)(\lambda - 1))} C_{33}^4 \\ \times \left[c^3(3\lambda^2 - 97) + c^2(94\lambda^3 - 467\lambda^2 - 483) + c(856\lambda^3 - 5192\lambda^2 + 4120) \right. \\ \left. + 216\lambda^3 - 6972\lambda^2 + 6756 \right].$$



Higher Structure Constants

Actually, can rewrite all of them more simply as

[MRG, Gopakumar]

$$C_{44}^4 = \frac{9(c+3)}{4(c+2)} \gamma - \frac{96(c+10)}{(5c+22)} \gamma^{-1}$$

$$(C_{34}^5)^2 = \frac{75(c+7)(5c+22)}{16(c+2)(7c+114)} \gamma^2 - 25$$

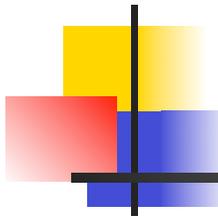
$$C_{45}^5 = \frac{15(17c+126)(c+7)}{8(7c+114)(c+2)} \gamma - 240 \frac{(c+10)}{(5c+22)} \gamma^{-1}$$

where

$$\gamma^2 \equiv (C_{33}^4)^2 = \frac{896}{75} \frac{N_4}{N_3^2}$$

Suggests that **all of these structure constants** are
determined by Jacobi identity.

[Candu, MRG, Kelm,
Vollenweider, to appear]



Quantum algebra

Thus full quantum algebra characterised by **two free parameters** [MRG, Gopakumar]

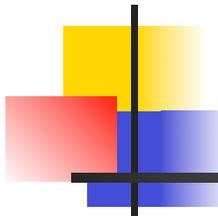
γ^2 and c .

But

$$(C_{33}^4)^2 \equiv \gamma^2 = \frac{64(c+2)(\lambda-3)(c(\lambda+3) + 2(4\lambda+3)(\lambda-1))}{(5c+22)(\lambda-2)(c(\lambda+2) + (3\lambda+2)(\lambda-1))}.$$

Thus there are **three roots** that lead to the **same algebra**:

$$\mathcal{W}_\infty[\lambda_1] \cong \mathcal{W}_\infty[\lambda_2] \cong \mathcal{W}_\infty[\lambda_3] \quad \text{at fixed } c$$



Triality

In particular,

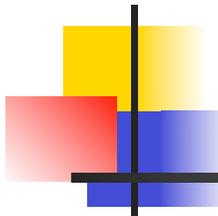
$$\mathcal{W}_\infty[N] \cong \mathcal{W}_\infty\left[\frac{N}{N+k}\right] \cong \mathcal{W}_\infty\left[-\frac{N}{N+k+1}\right] \quad \text{at } c = c_{N,k}$$

minimal model

asymptotic symmetry
algebra of hs theory

This is even true at finite N and k , not just in the 't Hooft limit!

This triality generalises level-rank duality of coset models of [Kuniba, Nakanishi, Suzuki] and [Altschuler, Bauer, Saleur].



Symmetries

So the symmetries suggest that we should have

HS on AdS3

=CS with
 $hs[\lambda]$



2d CFT with

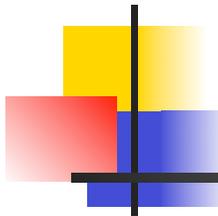
$\mathcal{W}_\infty[\lambda]$

symmetry

||

minimal models

Semiclassical limit: take c large --- 't Hooft limit!



Spectrum

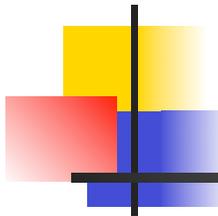
Higher spin fields themselves correspond only to the vacuum representation of the W-algebra!

To see this, calculate partition function of massless spin s field on thermal AdS3

[MRG, Gopakumar, Saha]

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \cdot \quad q = \exp\left(-\frac{1}{k_B T}\right)$$

[Generalisation of Giombi, Maloney & Yin calculation to higher spin, using techniques developed in David, MRG, Gopakumar.]



1-loop partition function

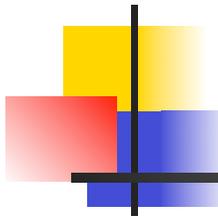
The **complete higher spin** theory therefore contributes

$$Z_{\text{hs}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \cdot$$

MacMahon
function!

This reproduces precisely contribution to the partition function of dual CFT in 't Hooft limit coming from the **vacuum representation**

--- not a consistent CFT by itself.....



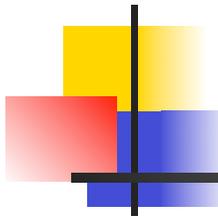
Representations

Indeed, the **full CFT** also has the representations labelled by (from coset description)

$$\begin{array}{ccc} & (\rho, \mu; \nu) & \\ \nearrow & \uparrow & \nwarrow \\ \text{rep of } \mathfrak{su}(N)_k & \mathfrak{su}(N)_1 & \mathfrak{su}(N)_{k+1} \end{array}$$

Compatibility constraint: $\rho + \mu - \nu \in \Lambda_R(\mathfrak{su}(N))$

fixes μ uniquely: **label representations** by $(\rho; \nu)$.



Simple representations

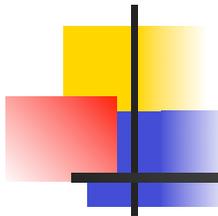
Simplest reps that **generate all W-algebra reps** upon fusion: $(0;f)$ and $(f;0)$ (& conjugates).

't Hooft limit: $h(f; 0) = \frac{1}{2}(1 + \lambda)$ $h(0; f) = \frac{1}{2}(1 - \lambda)$

semiclassical: $h(f; 0) = \frac{1}{2}(1 - N)$ $h(0; f) = -\frac{c}{2N^2}$

↑
dual to
perturbative
scalar

↑
non-perturbative



Proposal

Contribution from all representations of the form $(*;0)$ is accounted for by adding to the hs theory **a complex scalar field** of the **mass**

[MRG,Gopakumar]

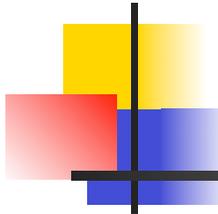
$$-1 \leq M^2 \leq 0 \quad \text{with} \quad M^2 = -(1 - \lambda^2) .$$

[Compatible with hs symmetry since hs theory has **massive scalar multiplet** with this mass.]

[Vasiliev]

Corresponding **conformal dimension** then

$$M^2 = \Delta(\Delta - 2) \Rightarrow \Delta = 1 + \lambda .$$



Checks of proposal

Main evidence from 1-loop calculation:

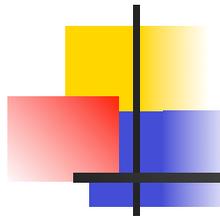
Contribution of single real scalar to thermal partition function is

[Giombi, Maloney & Yin]

$$Z_{\text{scalar}}^{(1)} = \prod_{l=0, l'=0}^{\infty} \frac{1}{(1 - q^{h+l} \bar{q}^{h+l'})} ,$$

where

$$h = \frac{1}{2} \Delta = \frac{1}{2} (1 + \lambda) .$$



Total 1-loop partition function

The total perturbative 1-loop partition function of our AdS theory is then

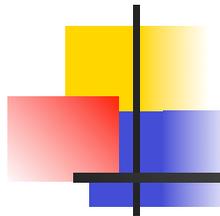
$$Z_{\text{pert}}^{(1)} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \times \prod_{l,l'=0}^{\infty} \frac{1}{(1 - q^{h+l} \bar{q}^{h+l'})^2}$$

We have shown analytically that this **agrees exactly with CFT partition function** of $(*;0)$ representations in 't Hooft limit!

[MRG,Gopakumar]

[MRG,Gopakumar,Hartman,Raju]

Strong consistency check!



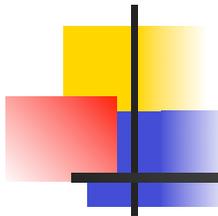
Non-perturbative states

The remaining states, i.e. those of the form

$$(*; \nu) \quad \text{with } \nu \neq 0$$

seem to correspond to **conical defect solutions**
(possibly dressed with perturbative excitations).

[Castro, Gopakumar, Gutperle, Raeymaekers]
[MRG, Gopakumar]



Generalisations

Various generalisations of the proposal have also been proposed and tested, in particular

- ▶ supersymmetric version

 - [Creutzig, Hikida, Ronne]

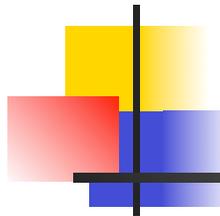
 - [Candu, MRG]

 - [Henneaux, Gomez, Park, Rey]

 - [Hanaki, Peng]

- ▶ orthogonal (instead of unitary) groups

 - [Ahn], [MRG, Vollenweider]



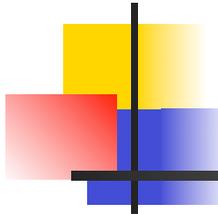
Classical solutions

Another very interesting development concerns the classical solutions of the HS theory.

[Gutperle, Kraus, et.al.]

Very **interesting lessons** (that are maybe applicable more generally): because of large HS gauge symmetry, usual **GR tensors are not gauge invariant** any longer!

Characterisation of regular classical solutions is therefore subtle!



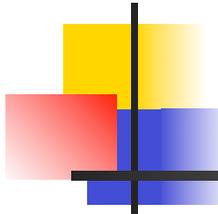
Black holes

However CS description allows for HS gauge invariant formulation. Using this point of view, **black hole solutions** for these theories have been **constructed**.

[Gutperle,Kraus,et.al.]

Their **entropy can be matched to dual CFT** description.

[Kraus,Perlmutter]
[MRG,Hartman,Jin]



Conclusions

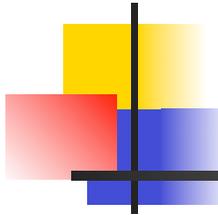
Given strong evidence for duality between

AdS3:
higher spin theory
with a complex
scalar of mass M



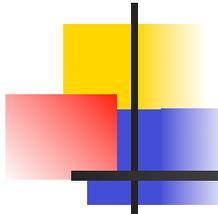
2d CFT:
 $\mathcal{W}_{N,k}$ minimal models
in large N 't Hooft limit
with coupling λ

where $\lambda = \frac{N}{N+k}$ and $M^2 = -(1 - \lambda^2)$



Conclusions

- ▶ The duality is **non-supersymmetric**.
- ▶ It allows for detailed **precision tests**: spectrum, correlation functions, etc.
- ▶ Can shed maybe interesting light on **conceptual aspects of quantum gravity**.



Main challenges

- ▶ Understand deformation of classical hs symmetry in quantum theory.

cf [Maldacena,Zhiboedov]

- ▶ Reproduce calculable quantum corrections of CFT from

Higher Spin Quantum Gravity
on AdS₃