

# Minimal Model Holography

---

Matthias Gaberdiel  
ETH Zürich

Ginzburg Conference on Physics  
1 June 2012

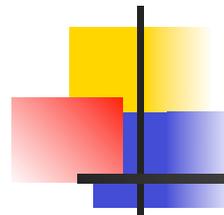
based mainly on

[MRG, R. Gopakumar](#), arXiv:1011.2986

[MRG, T. Hartman](#), arXiv:1101.2910

[MRG, R. Gopakumar, T. Hartman & S. Raju](#), arXiv:1106.1897

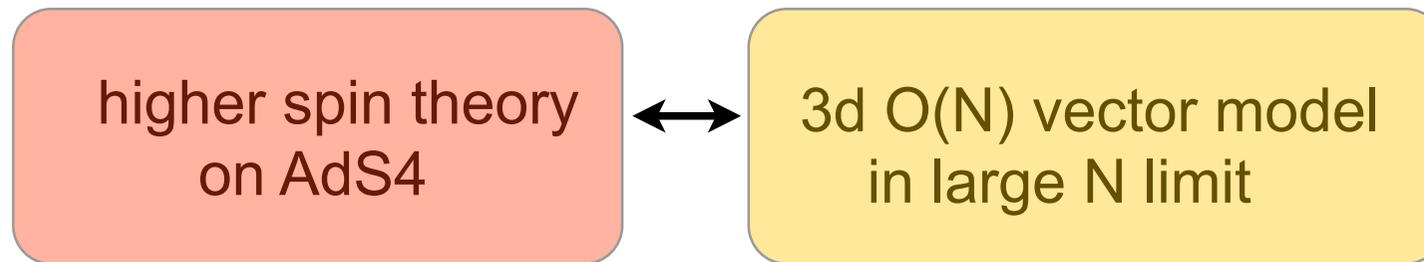
[MRG, R. Gopakumar](#), arXiv:1205.2472



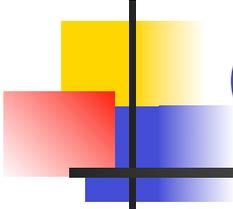
# Higher spin -- CFT duality

Some years ago, a concrete proposal for a higher spin - CFT duality was made:

[Klebanov-Polyakov]  
[Sezgin-Sundell]



Actually different versions, depending on whether vector model fields are **bosons or fermions** and on whether one considers **free or interacting fixed point**.



# Checks of the proposal

---

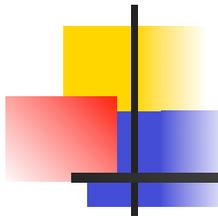
Recently **impressive checks** of the duality have been performed, in particular

3-point functions of HS fields on AdS4

have been **matched** to

3-point functions of HS currents in  $O(N)$  model to leading order in  $1/N$ .

[Giombi & Yin]



# AdS3 / CFT2

---

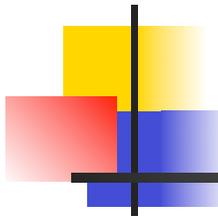
**Here:** describe 3d/2d CFT version of this duality.

Lower dimensional version interesting

- ▶ 2d CFTs well understood
- ▶ Higher spin theories simpler in 3d

Also, 3d conformal field theories with unbroken higher spin symmetry and finite number of d.o.f. (finite N) are necessarily free, but this is not the case in 2d.

[Maldacena,Zhiboedov]



# 3d proposal

---

The 3d/2d proposal takes the form

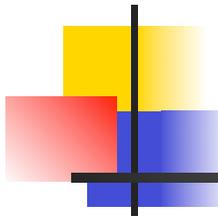
[MRG, Gopakumar]

**AdS3:**  
higher spin theory  
with a complex  
scalar of mass  $M$



**2d CFT:**  
 $\mathcal{W}_{N,k}$  minimal models  
in large  $N$  't Hooft limit  
with coupling  $\lambda$

where  $\lambda = \frac{N}{N+k}$  and  $M^2 = -(1 - \lambda^2)$



# Scalars

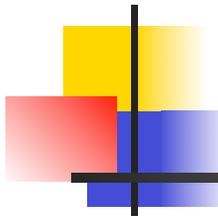
---

In original version of conjecture there were **two scalars**.

Given our more detailed understanding of the symmetries (see below), it now seems that **one of the scalars** should be rather thought of as a **non-perturbative** state.

[This new point of view resolves also some puzzles regarding the structure of the correlation functions.]

[Papadodimas, Raju]  
[Chang, Yin]



# Comparison to 4d/3d case

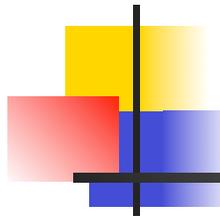
---

In contrast to 4d/3d case:

**1 parameter family** of dual theories.

Special values:

- ▶ For  $\lambda = 0$  the 2d CFT is equivalent to **singlet sector of a free theory**. [MRG,Suchanek]
- ▶ For  $\lambda = 1$  the resulting theory has linear  **$\mathcal{W}_\infty$  symmetry** (free bosons).

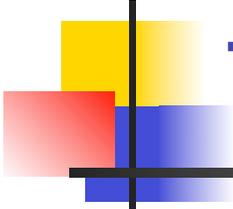


# Outline

---

In the rest of the talk I want to explain the proposal in more detail and indicate which consistency checks have been performed.

- The HS theory in 3d
- Matching the symmetries
- The spectrum
- Conclusions



# The HS theory on AdS3

---

The AdS3 HS theory can be described very simply.

Recall that pure gravity in AdS3: **Chern-Simons theory**  
based on

$$sl(2, \mathbb{R})$$

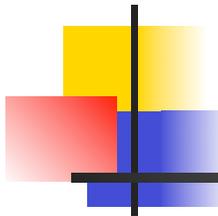
[Achucarro & Townsend]

[Witten]

Higher spin description: replace

$$sl(2, \mathbb{R}) \rightarrow \mathfrak{hs}[\lambda]$$

[Vasiliev]



# Higher spin algebra

---

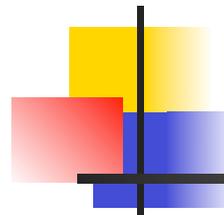
The higher spin algebra  $hs[\lambda]$  is an infinite dimensional Lie algebra that can be thought of as

$$hs[\lambda] \equiv sl(\lambda, \mathbb{R})$$

[Bordemann et.al.]  
[Bergshoeff et.al.]  
[Pope, Romans, Shen]  
[Fradkin, Linetsky]

since

$$hs[\lambda] \Big|_{\lambda=N} \cong sl(N, \mathbb{R}) \quad \text{for integer } N.$$



# Asymptotic symmetries

---

For these higher spin theories **asymptotic symmetry algebra** can be determined following **Brown & Henneaux**, leading to **classical**

$\mathcal{W}_\infty[\lambda]$  algebra

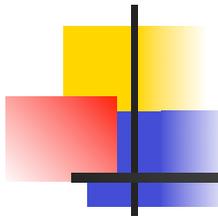
[Henneaux & Rey]  
[Campoleoni et al]  
[MRG, Hartman]

Extends algebra **'beyond the wedge'**:

pure gravity:  $sl(2, \mathbb{R}) \rightarrow$  Virasoro

higher spin:  $hs[\lambda] \rightarrow \mathcal{W}_\infty[\lambda]$

[Figueroa-O'Farrill et.al.]



# Dual CFT

---

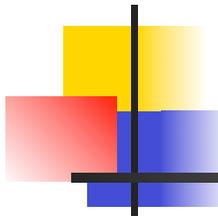
By the usual arguments, **dual CFT** should therefore have

$\mathcal{W}_\infty[\lambda]$  symmetry.

**Basic idea:**

$$\mathcal{W}_\infty[\lambda] = \lim_{N \rightarrow \infty} \mathcal{W}_N \quad \text{with} \quad \lambda = \frac{N}{N+k} .$$

't Hooft limit of 2d CFT!



# The minimal models

---

The minimal model CFTs are the **cosets**

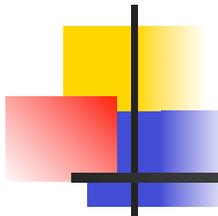
$$\mathcal{W}_{N,k} : \frac{su(N)_k \oplus su(N)_1}{su(N)_{k+1}}$$

←  
e.g. Ising model (N=2, k=1)  
tricritical Ising (N=2, k=2)  
3-state Potts (N=3, k=1),...

with **central charge**

$$c_N(k) = (N - 1) \left[ 1 - \frac{N(N + 1)}{(N + k)(N + k + 1)} \right].$$

General N: **higher spin analogue** of Virasoro minimal models. [Spin fields of spin  $s=2,3,\dots,N$ .]



# Relation of symmetries

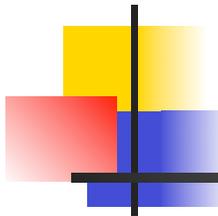
---

On the face of it, the two symmetries

$$\mathcal{W}_\infty[\lambda] \quad \text{vs} \quad \lim_{N,k \rightarrow \infty} \mathcal{W}_{N,k}$$

appear to be **quite different**. However, the asymptotic symmetry analysis only determines the **classical symmetry algebra**, i.e. the algebra to leading order in  $1/c$ .

In order to understand above relation, we need to understand the **quantum version of this algebra**.



# Quantum symmetry

---

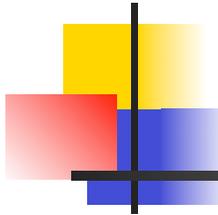
The full structure of the **quantum algebra** can actually be determined completely.

[MRG, Gopakumar]

There are **two steps** to this argument. To illustrate them consider an example. For **classical algebra**, we have

$$\begin{aligned}
 [W_m, W_n] = & 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\
 & + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}
 \end{aligned}$$

↑
↑  
 spin-3 field                      non-linear term



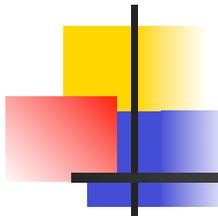
# Jacobi identity

---

$$[W_m, W_n] = 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3 c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

spin-3 field

non-linear term



# Jacobi identity

$$[W_m, W_n] = 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

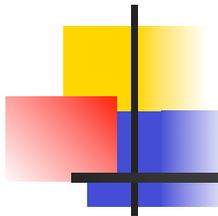
Jacobi identity determines **quantum correction**

$$[W_m, W_n] = 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ + \frac{8N_3}{c + \frac{22}{5}}(m-n)\Lambda_{m+n}^{(4)} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

where

$$\Lambda_n^{(4)} = \sum_p : L_{n-p}L_p : + \frac{1}{5}x_n L_n$$

Similar considerations apply for the other commutators.



# Structure constants

---

The **second step** concerns structure constants. W-field can be rescaled so that

$$W \cdot W \sim \frac{c}{3} \cdot \mathbf{1} + 2 \cdot L + \frac{32}{(5c + 22)} \cdot \Lambda^{(4)} + 4 \cdot U$$

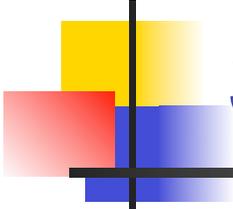
and

$$W \cdot U \sim \frac{56}{25} \frac{N_4}{N_3^2} W + \dots$$

spin-4 field

**Classical analysis** determines

$$\frac{N_4}{N_3^2} = \frac{15 \lambda^2 - 9}{14 \lambda^2 - 4} + \mathcal{O}\left(\frac{1}{c}\right).$$

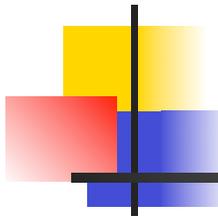


# Structure constants

---

Classical analysis determines

$$\frac{N_4}{N_3^2} = \frac{15 \lambda^2 - 9}{14 \lambda^2 - 4} + \mathcal{O}\left(\frac{1}{c}\right).$$



# Structure constants

---

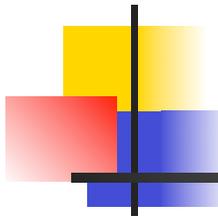
Classical analysis determines

$$\frac{N_4}{N_3^2} = \frac{15 \lambda^2 - 9}{14 \lambda^2 - 4} + \mathcal{O}\left(\frac{1}{c}\right).$$

Requirement that representation theory agrees for  $\lambda = N$  with  $\mathcal{W}_N$  :

$$\frac{N_4}{N_3^2} = \frac{75 (c + 2) (\lambda - 3) (c(\lambda + 3) + 2(4\lambda + 3)(\lambda - 1))}{14 (5c + 22) (\lambda - 2) (c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}.$$

[Note:  $\text{hs}[\lambda] \Big|_{\lambda=N} \cong \mathfrak{sl}(N, \mathbb{R})$  implies  $\mathcal{W}_\infty[\lambda] \Big|_{\lambda=N} = \mathcal{W}_N$  .]



# Higher Structure Constants

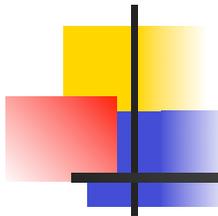
Similarly, **higher structure constants** can be determined

[Blumenhagen, et.al.] [Hornfeck]

$$C_{33}^4 C_{44}^4 = \frac{48(c^2(\lambda^2 - 19) + 3c(6\lambda^3 - 25\lambda^2 + 15) + 2(\lambda - 1)(6\lambda^2 - 41\lambda - 41))}{(\lambda - 2)(5c + 22)(c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}$$

$$(C_{34}^5)^2 = \frac{25(5c + 22)(\lambda - 4)(c(\lambda + 4) + 3(5\lambda + 4)(\lambda - 1))}{(7c + 114)(\lambda - 2)(c(\lambda + 2) + (3\lambda + 2)(\lambda - 1))}$$

$$C_{45}^5 = \frac{15}{8(\lambda - 3)(c + 2)(114 + 7c)(c(\mu + 3) + 2(4\lambda + 3)(\lambda - 1))} C_{33}^4 \\ \times \left[ c^3(3\lambda^2 - 97) + c^2(94\lambda^3 - 467\lambda^2 - 483) + c(856\lambda^3 - 5192\lambda^2 + 4120) \right. \\ \left. + 216\lambda^3 - 6972\lambda^2 + 6756 \right].$$



# Higher Structure Constants

Actually, can rewrite all of them more simply as

[MRG, Gopakumar]

$$C_{44}^4 = \frac{9(c+3)}{4(c+2)} \gamma - \frac{96(c+10)}{(5c+22)} \gamma^{-1}$$

$$(C_{34}^5)^2 = \frac{75(c+7)(5c+22)}{16(c+2)(7c+114)} \gamma^2 - 25$$

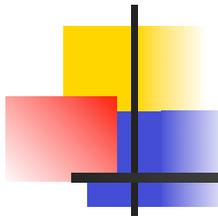
$$C_{45}^5 = \frac{15(17c+126)(c+7)}{8(7c+114)(c+2)} \gamma - 240 \frac{(c+10)}{(5c+22)} \gamma^{-1}$$

where

$$\gamma^2 \equiv (C_{33}^4)^2 = \frac{896}{75} \frac{N_4}{N_3^2}$$

Suggests that **all of these structure constants** are  
determined by Jacobi identity.

[Candu, MRG, Kelm,  
Vollenweider, to appear]



# Quantum algebra

---

Thus full quantum algebra characterised by **two free parameters** [MRG, Gopakumar]

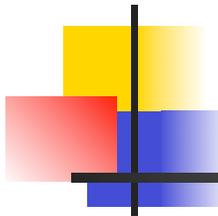
$\gamma^2$  and  $c$ .

But

$$(C_{33}^4)^2 \equiv \gamma^2 = \frac{64(c+2)(\lambda-3)(c(\lambda+3) + 2(4\lambda+3)(\lambda-1))}{(5c+22)(\lambda-2)(c(\lambda+2) + (3\lambda+2)(\lambda-1))}.$$

Thus there are **three roots** that lead to the **same algebra**:

$$\mathcal{W}_\infty[\lambda_1] \cong \mathcal{W}_\infty[\lambda_2] \cong \mathcal{W}_\infty[\lambda_3] \quad \text{at fixed } c$$



# Triality

---

In particular,

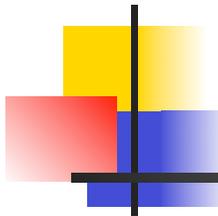
$$\mathcal{W}_\infty[N] \cong \mathcal{W}_\infty\left[\frac{N}{N+k}\right] \cong \mathcal{W}_\infty\left[-\frac{N}{N+k+1}\right] \quad \text{at } c = c_{N,k}$$

minimal model

asymptotic symmetry  
algebra of hs theory

This is even true at finite  $N$  and  $k$ , not just in the 't Hooft limit!

This triality generalises level-rank duality of coset models of [Kuniba, Nakanishi, Suzuki] and [Altschuler, Bauer, Saleur].



# Symmetries

---

So the symmetries suggest that we should have

HS on AdS3

=CS with  
 $hs[\lambda]$



2d CFT with

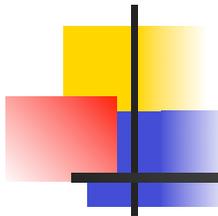
$\mathcal{W}_\infty[\lambda]$

symmetry

||

minimal models

Semiclassical limit: take  $c$  large --- 't Hooft limit!



# Spectrum

---

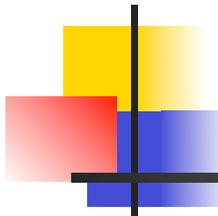
Higher spin fields themselves correspond only to the vacuum representation of the W-algebra!

To see this, calculate partition function of massless spin  $s$  field on thermal AdS3

[MRG, Gopakumar, Saha]

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \cdot \quad q = \exp\left(-\frac{1}{k_B T}\right)$$

[Generalisation of Giombi, Maloney & Yin calculation to higher spin, using techniques developed in David, MRG, Gopakumar.]



# 1-loop partition function

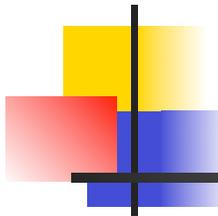
The **complete higher spin** theory therefore contributes

$$Z_{\text{hs}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \cdot$$

MacMahon  
function!

This reproduces precisely contribution to the partition function of dual CFT in 't Hooft limit coming from the **vacuum representation**

--- not a consistent CFT by itself.....



# Representations

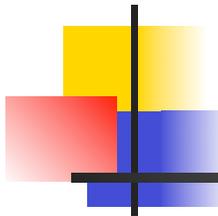
---

Indeed, the **full CFT** also has the representations labelled by (from coset description)

$$\begin{array}{ccc} & (\rho, \mu; \nu) & \\ \nearrow & \uparrow & \nwarrow \\ \text{rep of } \mathfrak{su}(N)_k & \mathfrak{su}(N)_1 & \mathfrak{su}(N)_{k+1} \end{array}$$

**Compatibility constraint:**  $\rho + \mu - \nu \in \Lambda_R(\mathfrak{su}(N))$

fixes  $\mu$  uniquely: **label representations** by  $(\rho; \nu)$  .



# Simple representations

---

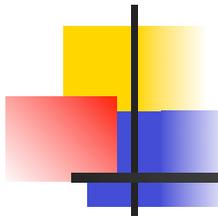
Simplest reps that generate all W-algebra reps upon fusion:  $(0;f)$  and  $(f;0)$  (& conjugates).

't Hooft limit:  $h(f; 0) = \frac{1}{2}(1 + \lambda)$        $h(0; f) = \frac{1}{2}(1 - \lambda)$

semiclassical:  $h(f; 0) = \frac{1}{2}(1 - N)$        $h(0; f) = -\frac{c}{2N^2}$

↑  
dual to  
perturbative  
scalar

↑  
non-perturbative



# Proposal

---

Contribution from all representations of the form  $(*;0)$  is accounted for by adding to the hs theory **a complex scalar field** of the **mass**

[MRG,Gopakumar]

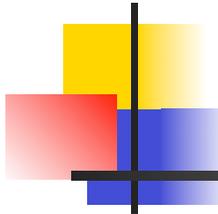
$$-1 \leq M^2 \leq 0 \quad \text{with} \quad M^2 = -(1 - \lambda^2) .$$

[Compatible with hs symmetry since hs theory has **massive scalar multiplet** with this mass.]

[Vasiliev]

Corresponding **conformal dimension** then

$$M^2 = \Delta(\Delta - 2) \Rightarrow \Delta = 1 + \lambda .$$



# Checks of proposal

---

**Main evidence** from 1-loop calculation:

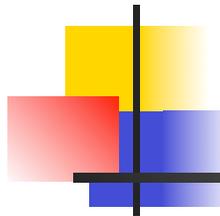
Contribution of single real scalar to thermal partition function is

[Giombi, Maloney & Yin]

$$Z_{\text{scalar}}^{(1)} = \prod_{l=0, l'=0}^{\infty} \frac{1}{(1 - q^{h+l} \bar{q}^{h+l'})} ,$$

where

$$h = \frac{1}{2} \Delta = \frac{1}{2} (1 + \lambda) .$$



# Total 1-loop partition function

---

The total perturbative 1-loop partition function of our AdS theory is then

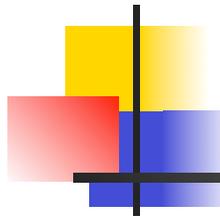
$$Z_{\text{pert}}^{(1)} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} \times \prod_{l,l'=0}^{\infty} \frac{1}{(1 - q^{h+l} \bar{q}^{h+l'})^2}$$

We have shown analytically that this **agrees exactly with CFT partition function** of  $(*;0)$  representations in 't Hooft limit!

[MRG,Gopakumar]

[MRG,Gopakumar,Hartman,Raju]

**Strong consistency check!**



# Non-perturbative states

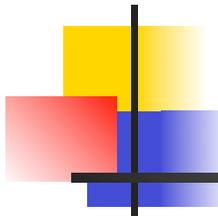
---

The remaining states, i.e. those of the form

$$(*; \nu) \quad \text{with } \nu \neq 0$$

seem to correspond to **conical defect solutions**  
(possibly dressed with perturbative excitations).

[Castro, Gopakumar, Gutperle, Raeymaekers]  
[MRG, Gopakumar]



# Generalisations

---

Various generalisations of the proposal have also been proposed and tested, in particular

- ▶ supersymmetric version

  - [Creutzig, Hikida, Ronne]

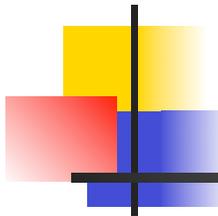
  - [Candu, MRG]

  - [Henneaux, Gomez, Park, Rey]

  - [Hanaki, Peng]

- ▶ orthogonal (instead of unitary) groups

  - [Ahn], [MRG, Vollenweider]



# Classical solutions

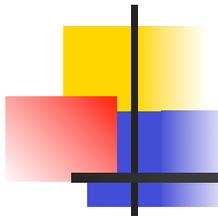
---

Another very interesting development concerns the classical solutions of the HS theory.

[Gutperle, Kraus, et.al.]

Very **interesting lessons** (that are maybe applicable more generally): because of large HS gauge symmetry, usual **GR tensors are not gauge invariant** any longer!

Characterisation of regular classical solutions is therefore subtle!



# Black holes

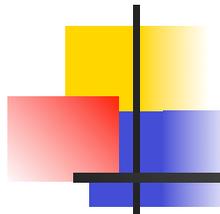
---

However CS description allows for HS gauge invariant formulation. Using this point of view, **black hole solutions** for these theories have been **constructed**.

[Gutperle,Kraus,et.al.]

Their **entropy can be matched to dual CFT description**.

[Kraus,Perlmutter]  
[MRG,Hartman,Jin]



# Conclusions

---

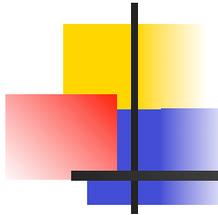
Given strong evidence for duality between

**AdS3:**  
higher spin theory  
with a complex  
scalar of mass  $M$



**2d CFT:**  
 $\mathcal{W}_{N,k}$  minimal models  
in large  $N$  't Hooft limit  
with coupling  $\lambda$

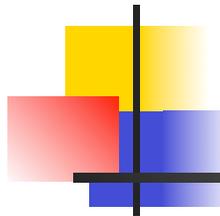
where  $\lambda = \frac{N}{N+k}$  and  $M^2 = -(1 - \lambda^2)$



# Conclusions

---

- ▶ The duality is **non-supersymmetric**.
- ▶ It allows for detailed **precision tests**: spectrum, correlation functions, etc.
- ▶ Can shed maybe interesting light on **conceptual aspects of quantum gravity**.



# Main challenges

---

- ▶ Understand deformation of classical hs symmetry in quantum theory.

cf [Maldacena,Zhiboedov]

- ▶ Reproduce calculable quantum corrections of CFT from

Higher Spin Quantum Gravity  
on AdS<sub>3</sub>