Wilson loop remainder function for null polygons in the limit of self-crossing

work with Sebastian Wuttke : arXiv:1104.2469 and 1111.6815

- Motivation and introduction
- Strategy and results
- Comparison with full analytic results
- Some details of the calculation
- Conclusions

Study Wilson loops in $\mathcal{N} = 4$ SYM in the planar limit for special contours:

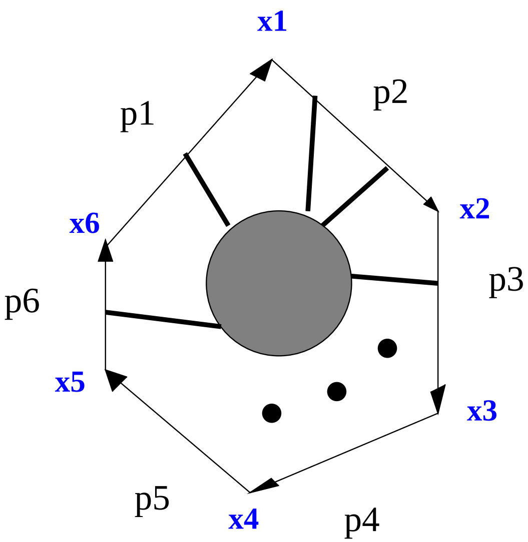
Null polygons with vertices $x_1,\ x_2, \ldots,\ x_n$ and

$$\text{edges } p_k = x_k - x_{k-1}, \quad k = 1, \ldots,\ n$$

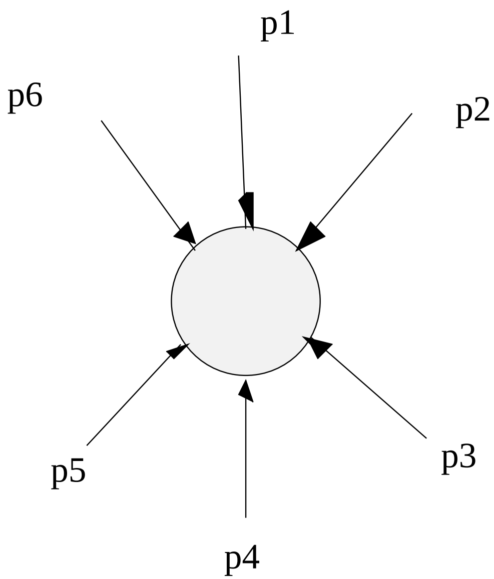
These objects are of interest in its own and with respect to the correspondence between

Wilson loops, MHV gluon scattering amplitudes and string surfaces in AdS

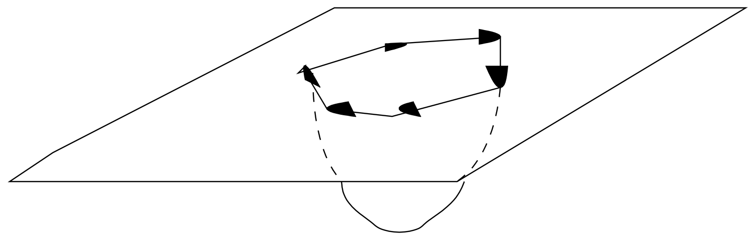
Wilson loops UV divergent, scattering amplitudes IR divergent, in dimensional regularisation $\epsilon_{IR} \Leftrightarrow -\epsilon_{UV}$



Wilson loop for null hexagon



6 point scattering amplitude



string surface in AdS

$$\log W \;=\; \sum_l \; a^l \Big(f^{(l)}(\epsilon) W^{(1)}(l\epsilon, \{s\}) + C^{(l)}(\epsilon) \Big) + R(a, \{u\}) + \mathcal{O}(\epsilon) \;,$$

with $\{s\}$ the set of Mandelstam variables $s_{kl} = (x_k - x_l)^2$ and

$\{u\}$ the conformally invariant cross ratios form out of the s_{kl}.

$a = \frac{g^2 N}{8\pi^2}$, $\quad W^{(1)}(\epsilon, \{s\})$ one loop contribution.

The blue part is the BDS structure, fixed also by anomalous conformal Ward identities.

R is called the remainder function,

it appears for $n \geq 6$ only and starts at order a^2.

Remarkable: Single diagrammatic contrib. to $\log W$ (via non-Abelian expo theorem) contain higher powers $\frac{1}{\epsilon^k}$, $k > 2$

but $\log W$ has only $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$.

Situation changes if contour has self-crossing ($R = a^2\, R^{(2)} + a^3\, R^{(3)} + \ldots$)

$R^{(2)} \propto \frac{1}{\epsilon^3} + \ldots\,,\qquad R^{(3)} \propto \frac{1}{\epsilon^5} + \ldots$

(From now: $R := \log W -$ BDS also for $\epsilon \neq 0$.)

⇒ interesting problem in its own

⇒ alternative aspect

Approach to self-crossing from a generic configuration:

The (generically) finite R develops singularities

$\propto\ \log^3(1-u)$ for $R^{(2)}$

$\propto\ \log^5(1-u)$ for $R^{(3)}$

if some characteristic cross-ratio $u \rightarrow 1$.

Available info on R in generic configuration:

hexagon $R^{(2)}$: full analytic result Goncharov, Spradlin, Vergu, Volovich 2010

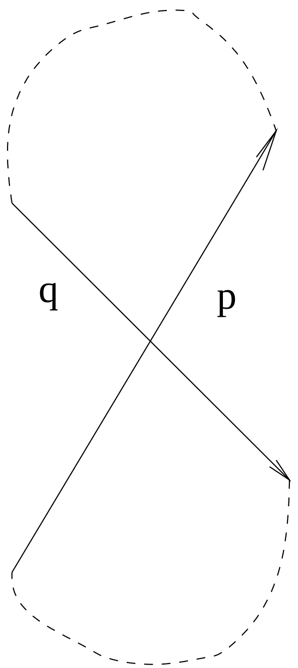
hexagon $R^{(3)}$: full symbol Dixon, Drummond, Henn and Caron-Huot, He 2011

$\exists$ also info on symbols for higher polygons and/or explicit analytic results for restricted configurations (2D).

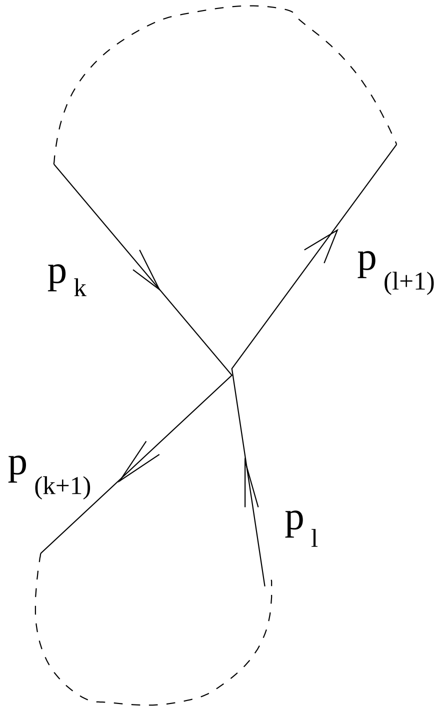
Different self-crossing types:

a) crossing of two edges, not ∃ free conformally invariant parameter

b) touching of two vertices, ∃ one free conformally invariant parameter



a)



b)

$R^{(2)}$ for case of touching vertices studied in our 1104.2469

This talk: Concentrate on crossing edge case, $R^{(2)}$, $R^{(3)}$ 1111.6815

Related singularities for scattering amplitudes:

touching vertices $\Leftrightarrow$ momentum conservation for a subset of momenta

crossing edges $\Leftrightarrow$ $\exists$ two opposite external momenta whose both adjacent inner momenta become in some integration region collinear to them,

might be relevant for double parton scattering,

e.g. Gaunt, Stirling 2011

Use RG-equation for self-crossing Wilson loops

& input $\log W = \text{BDS} + R$ Georgiou 2009

- RG-equation with mixing under renormalisation, due to self-crossing
- Use general structure of anomalous dimension matrix

  in case of <u>null</u>-edges Korchemskaya, Korchemsky 1994
- Book-keep the dependence on $\log \mu^2$, μ RG-scale
- Concentrate on leading and nextleading power of $\log \mu^2$
- μ enters only via $a \to a\mu^{2\epsilon}$,

  hence one can reconstruct corresponding poles in ϵ for $R(\epsilon, \mu, \{s\})$

$$R^{(2)} = \frac{i\pi}{4}\Big(\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\, 2\log(2pq\mu^2\mathcal{X})\Big) + \mathcal{O}(\tfrac{1}{\epsilon}) , \qquad pq > 0 ,$$

$$R^{(2)} = -\frac{i\pi}{4}\Big(\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\, 2\log(2|pq|\mu^2\mathcal{X})\Big) + \frac{\pi^2}{2}\,\frac{1}{\epsilon^2} + \mathcal{O}(\tfrac{1}{\epsilon}) , \qquad pq < 0 .$$

$$R^{(3)} = -\frac{7i\pi}{108}\Big(\frac{1}{\epsilon^5} + \frac{1}{\epsilon^4}\, 3\log(2pq\mu^2\mathcal{X})\Big) - \frac{\pi^2}{18}\,\frac{1}{\epsilon^4} + \mathcal{O}(\tfrac{1}{\epsilon^3}) , \qquad pq > 0 ,$$

$$R^{(3)} = \frac{7i\pi}{108}\Big(\frac{1}{\epsilon^5} + \frac{1}{\epsilon^4}\, 3\log(2|pq|\mu^2\mathcal{X})\Big) - \frac{\pi^2}{4}\,\frac{1}{\epsilon^4} + \mathcal{O}(\tfrac{1}{\epsilon^3}) , \qquad pq < 0 .$$

$\mathcal{X} = xy(1-x)(1-y)$, x, y fractions on edges p and q defining the crossing point.

Note: coefficient of nextleading pole is not conformally invariant,

OK since at $\epsilon \neq 0$ conformal invariance broken.

Relation to singularities of generic (i.e. no self-crossing) $R(a,\{u\})$

for some special $u_j \to 1$:

- Consider conf. slightly off self-crossing as an alternative regularisation: distance $z_\perp$
- Argue (heuristically) for a "translation rule"

$$g^{2l}\,\frac{1}{\epsilon^m} \;\Leftrightarrow\; \alpha_{l,m}\, g^{2l} \log^m\!\left(\frac{1}{-\mu^2 z_\perp^2}\right), \quad \alpha_{l,m} = \frac{l^{m-l}\; l!}{m!}, \quad (\text{note:} \quad \alpha_{l,l} = 1)$$

- Pure geometry of near self-crossing:

$$\begin{aligned} \log\left(\frac{1}{-\mu^2 z_\perp^2}\right) &= -\log(u-1) - \log(-2pq\mu^2\mathcal{X}) + \mathcal{O}(z_\perp^2), \quad pq < 0 \\ &= -\log(1-u) - \log(2pq\mu^2\mathcal{X}) + \mathcal{O}(z_\perp^2), \quad pq > 0 \end{aligned}$$

u cross ratio formed out of the 4 endpoints of the crossing edges.

$$\begin{aligned} R^{(2)} &= \frac{i\pi}{6}\log^3(u-1) + \frac{\pi^2}{2}\log^2(u-1) + \mathcal{O}(\log(u-1))\,, \quad pq < 0 \\ &= -\frac{i\pi}{6}\log^3(1-u) + \mathcal{O}(\log(1-u))\,, \quad pq > 0 \end{aligned}$$

$$R^{(3)} = -\frac{7}{240}\, i\pi\, \log^5(u-1) - \frac{3}{16}\pi^2 \log^4(u-1) + \mathcal{O}(\log^3(u-1))\,, \quad pq < 0$$

$$= \frac{7}{240}\, i\pi\, \log^5(1-u) - \frac{1}{24}\pi^2 \log^4(1-u) + \mathcal{O}(\log^3(1-u))\,, \quad pq > 0$$

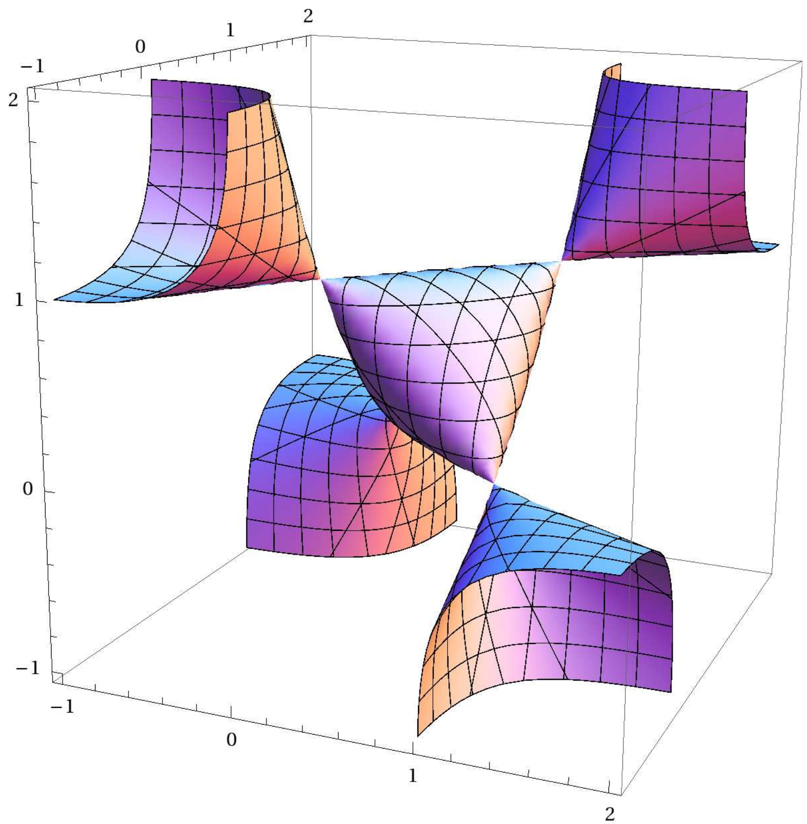
- For $R^{(2)}$ full agreement with corresponding limit for result of Goncharov, Spradlin, Vergu, Volovich
- $R^{(3)}$: disagreement by factor $\frac{6}{7}$ with leading singularity from symbolic result of Dixon, Drummond, Henn

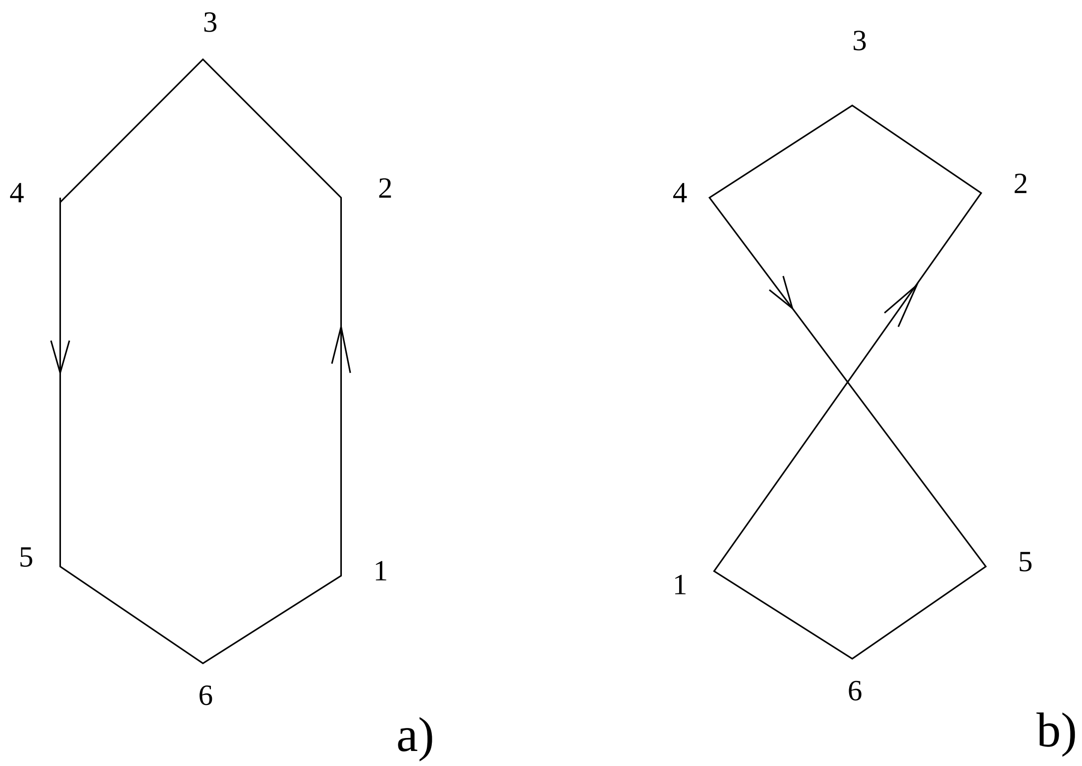
Comparison with full analytic result

$$R^{(2)} = -\frac{1}{2}\,\mathrm{Li}_4(1-\tfrac{1}{u}) - \frac{1}{8}\left(\mathrm{Li}_2(1-\tfrac{1}{u})\right)^2 + \ \ldots\,, \qquad \text{GSVV}$$

derived in Euclidean region, at first sight no singularity at $u \to 1$.

But: The three independent cross-ratios u_1, u_2, u_3 do <u>not</u> fix the conformal class of hexagon configurations.





projection on (1,2)-plane, edges running backward and forward in time, both a) and b) have $u_2 = 1$.

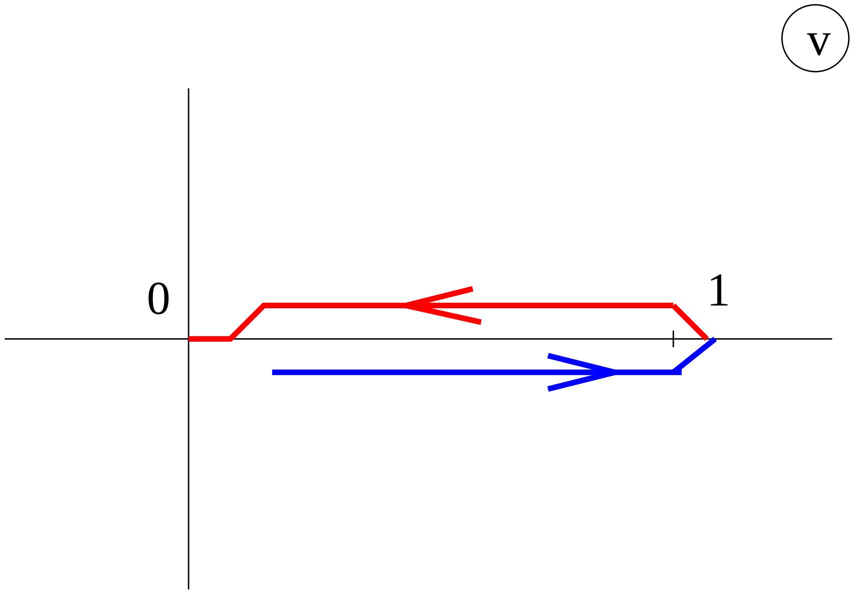
In twisting a) to b) $\frac{1}{u_2}$ goes from 1 to 0 and back to 1, i.e. argument of Polylogs $v := 1 - \frac{1}{u}$: $v = 0 \longrightarrow v = 1 \longrightarrow v = 0$

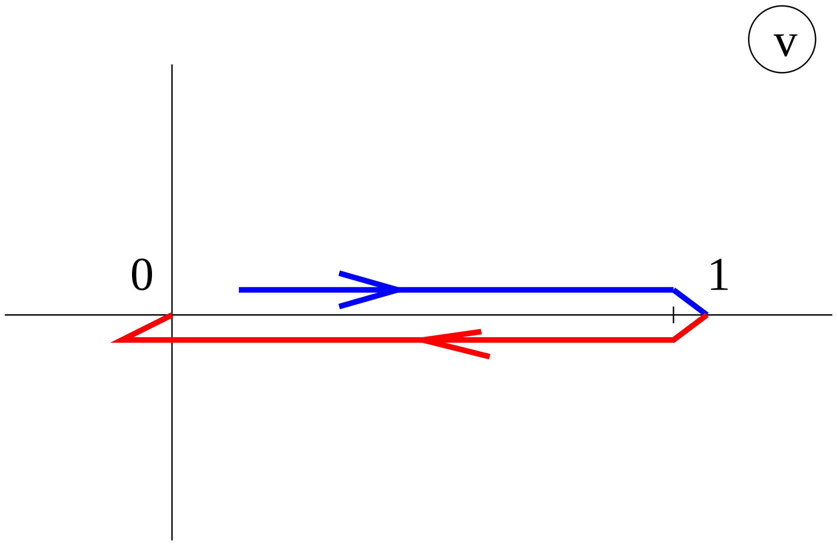
Implementing the $i\varepsilon$-prescription $\Rightarrow$ "reflection" at $v = 1$ (branchpoint of the Li's) is combined with encircling

Twisting moves us into second sheet of the Polylogs, there we hit logarithmic singularities at $v = 0$, i.e. $u = 1$.

$$\mathrm{Li}_n(v + i\varepsilon) - \mathrm{Li}_n(v - i\varepsilon) \; = \; \frac{2\pi i \; \log^{n-1} v}{(n-1)!}$$

Comparison with full analytic result





pq < 0,   u −−> 1+0,   v −−> +0

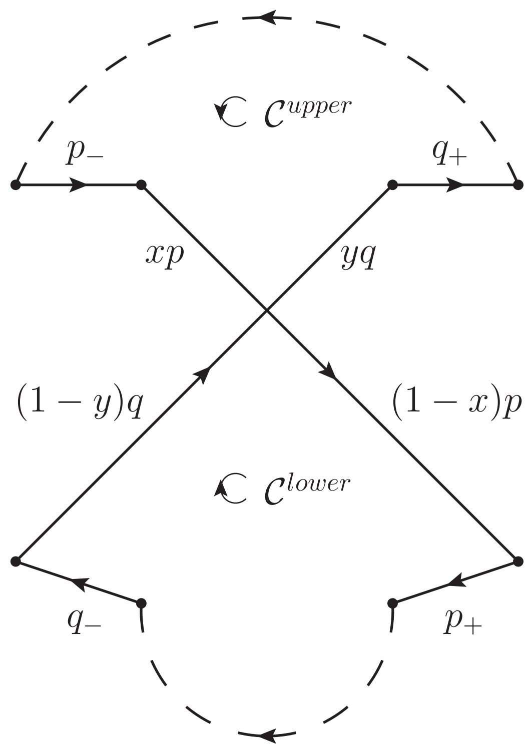
pq > 0, u −−> 1−0,   v −−> −0

$$W_a \;=\; Z_{ab}\, W_b^{\text{ren}}\ , \quad W_1 := \langle U(\mathcal{C})\rangle\ , \quad W_2 := \langle U(\mathcal{C}^{\text{upper}})U(\mathcal{C}^{\text{lower}})\rangle\ ,$$

$$U(\mathcal{C}) := \frac{1}{N}\ \text{tr}\ \mathcal{P}\ \ \text{exp}\,(ig\textstyle\int_{\mathcal{C}} A^{\mu}\text{d}x_{\mu})\ , \qquad \Gamma := Z^{-1}\mu\frac{d}{d\mu}Z\,\Big|_{g_{\text{bare}}\ \text{fixed}}\ ,$$

$$\mu\frac{\partial}{\partial\mu}\ \log W_1^{\text{ren}} \;=\; -\ \Gamma_{12}\,\frac{W_2^{\text{ren}}}{W_1^{\text{ren}}}\ -\ \Gamma_{11}\ ,$$

β-function zero !!



Anomalous dim. due to cusps and self-crossings for time-like or space-like contours depend on angles ϑ, $\cosh\vartheta = \frac{pq}{\sqrt{p^2q^2}}$, $\vartheta \to \infty$ for $p^2, q^2 \to 0$.

Div. linear in ϑ to all orders, modified type of RG-equation with anomalous dimensions depending linearly on $\log\mu$ (μ RG-scale). Korchemsky

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\Gamma_{\text{cusp}}(a)}{2} \sum_{k \in \text{cusps, not adj. crossing}} \log\left(-s_k\mu^2\right) + \begin{pmatrix} A & \gamma_{12}(a) \\ 0 & B \end{pmatrix} ,$$

$$\begin{aligned} A =& \frac{\Gamma_{\text{cusp}}(a)}{2} \left(\log\left(-2pp_-\mu^2\right) + \log\left(-2pp_+\mu^2\right) + \log\left(-2qq_-\mu^2\right) + \log\left(-2qq_+\mu^2\right)\right) , \\ B =& \frac{\Gamma_{\text{cusp}}(a)}{2} \left(\log\left(-2pp_-x\mu^2\right) + \log\left(-2pp_+(1-x)\mu^2\right) + \log\left(-2qq_-(1-y)\mu^2\right)\right. \\ & \left. + \log\left(-2qq_+y\mu^2\right)\right) \\ & + \gamma_{22}(a) \left(\log\left(-sxy\mu^2\right) + \log\left(-s(1-x)(1-y)\mu^2\right)\right) , \end{aligned}$$

planar approximation, $s = 2pq$, $s_k = (x_{k+1} - x_{k-1})^2$.

$Z \Leftrightarrow \Gamma$ relation:

$$\mu \frac{d}{d\mu} \log Z_{11} = \Gamma_{11} \,, \qquad \mu \frac{d}{d\mu} Z_{12} = Z_{11}\Gamma_{12} + Z_{12}\Gamma_{22} \,,$$

$$\mu \frac{d}{d\mu} \log Z_{22} = \Gamma_{22} \,, \qquad \text{with} \quad \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} - 2\epsilon \, a \, \frac{\partial}{\partial a}$$

$$Z_{11}^{(0)} = 1 \,, \qquad Z_{11}^{(1)} = -\frac{n\Gamma_{\rm cusp}^{(1)}}{4\epsilon^2} - \frac{\Gamma_{11}^{(1)}}{2\epsilon} \,, \qquad Z_{12}^{(1)} = -\frac{\gamma_{12}^{(1)}}{2\epsilon},$$

$$Z_{22}^{(0)} = 1 \,, \qquad Z_{22}^{(1)} = -\frac{n\Gamma_{\rm cusp}^{(1)} + 4\gamma_{22}^{(1)}}{4\epsilon^2} - \frac{\Gamma_{22}^{(1)}}{2\epsilon} \,.$$

$$Z_{12}^{(2)} = \frac{(2n+1)\gamma_{12}^{(1)}}{8\epsilon^3} + \frac{\gamma_{12}^{(1)}\left(\Gamma_{11}^{(1)} + \Gamma_{22}^{(1)}\right)}{8\epsilon^2} - \frac{\gamma_{12}^{(2)}}{4\epsilon} \,.$$

Then from $W_j \;=\; Z_{jk}\, W_k^{\text{ren}}$ and expanding $\log W_j$ in powers of a

$$
\begin{aligned}
(\log W_1^{\text{ren}})^{(1)} &= W_1^{\text{ren}(1)} \;=\; \text{MS}\Big[(\log W_1)^{(1)}\Big]\;, \\
(\log W_1^{\text{ren}})^{(2)} &= \text{MS}\Big[(\log W_1)^{(2)} + Z_{12}^{(1)}\left(W_1^{\text{ren}(1)} - W_2^{\text{ren}(1)}\right)\Big]\;, \\
(\log W_1^{\text{ren}})^{(3)} &= \text{MS}\Big[(\log W_1)^{(3)} \;-\; T_1 \;-\; T_2\Big]\;,
\end{aligned}
$$

with $\text{MS}\big[\;\ldots\;\big]$ denoting minimal subtraction and

$$
\begin{aligned}
T_1 &:= Z_{12}^{(1)}\left(\frac{1}{2}\left(W_1^{\text{ren}(1)} - W_2^{\text{ren}(1)}\right)^2 - (\log W_1^{\text{ren}})^{(2)} + (\log W_2^{\text{ren}})^{(2)}\right)\;, \\
T_2 &:= \left(\left(Z_{12}^{(1)}\right)^2 + Z_{12}^{(1)} Z_{11}^{(1)} - Z_{12}^{(2)}\right)\left(W_1^{\text{ren}(1)} - W_2^{\text{ren}(1)}\right)\;.
\end{aligned}
$$

- We have under control the dependence on $L := \log(\mu^2)$ of:
  - $(\log W_1^{\text{ren}})^{(1)}$, $(\log W_2^{\text{ren}})^{(1)}$: up to L^2
  - $(\log W_1^{\text{ren}})^{(2)}$, $(\log W_2^{\text{ren}})^{(2)}$: up to L^3
  - the BDS contribution to $(\log W_1^{\text{ren}})^{(3)}$: up to L^2
- due to poles up to $\frac{1}{\epsilon^3}$ in the Z-factors, vanishing terms up to $\mathcal{O}(\epsilon^3)$ from $\left(W_1^{\text{ren}(1)} - W_2^{\text{ren}(1)}\right)$ are relevant,

  and contribute up to L^5

Together we find:

$$(\log W_1^{\text{ren}})^{(3)} = \text{MS}\Big[(\log W_1)^{(3)}\Big] + \text{number} \cdot L^5 + \text{number} \cdot L^4 + \mathcal{O}(L^3) \,.$$

Inserting this into the starting RG-equation for $\log W_1^{\text{ren}}$ and book-keep order a^3 one gets

the leading and nextleading (L^5 and L^4) dependence of MS$\left[R^{(3)}\right]$.

- In case of two crossing edges leading and nextleading UV divergence of remainder determined by one-loop info on anomalous dimensions
- Explicit results for $R^{(2)}$ and $R^{(3)}$ in dimensional regularisation
- Treatment of higher orders seems realistic
- Studied $R^{(2)}$ also in case of touching vertices, here already leading divergence requires two-loop info on anomalous dimensions
- Translation into singularities of generic remainder for the approach to self-crossing can give checks and hints for the search to
  full analytic results, similar to multi Regge limit, collinear limit, ...
- Heuristic translation rule works perfect up to two loops, gives correct <u>relative</u> weight at three loops (ensuring conformal invariance)

- Factor 6/7 discrepancy relative to info from symbolic results in literature
- Work in progress: direct analysis of Feynman diagrams