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Fundamental Spinor Quantum Gravity regularized on a lattice

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D.D., A.Tumanov, A. Vladimirov, Phys. Rev. D84, 124042 (2011)

D.D., arXiv:1109.0091; A. Vladimirov and D.D., in preparation

The logic

- Fermions ⇒ Cartan (not Riemann) geometry: torsion is generally nonzero
- Classically, torsion turns out to be zero, the observational difference between two formulations is undistinguishable
- Quantum mechanically, though, in Cartan formulation large fluctuations of metrics are not restricted, as a matter of principle
- A way out: Spinor Quantum Gravity, where the tetrad is a bilinear fermion “current”, and looks like the Standard Model
- Spinor quantum gravity is easily regularized on a diffeomorphism-invariant lattice. It is a well-defined and well-behaved quantum theory
- Spinor quantum gravity typically breaks chiral symmetry, or fermion number conservation
- Presumably we “live” at the phase transition point, which guarantees long-range gravity

Fermions in General Relativity

There are fermions in Nature that need to be incorporated into the GR.
The standard way is by V. Fock and H. Weyl (1929): it involves new entities that
are **not** encountered in Riemann geometry – the ***frame field*** and the ***spin connection***.

$$S_{\mathrm{f}} = i \int d^4x \, \det(e) \, \frac{1}{2} \left(\overline{\Psi} \, e^{A\mu} \gamma_A \mathcal{D}_\mu \Psi - \overline{\mathcal{D}_\mu \Psi} \, e^{A\mu} \gamma_A \Psi \right), \qquad \mathcal{D}_\mu = \partial_\mu + \frac{1}{8} \omega_\mu^{BC} [\gamma_B \gamma_C]$$

$$\det(e) \, e^{D\nu} = \frac{1}{6} \epsilon^{\kappa\lambda\mu\nu} \, \epsilon^{ABCD} \, e_\kappa^A e_\lambda^B e_\mu^C, \qquad e^{D\nu} e_{E\nu} = \delta_E^D.$$

the contravariant tetrad is the inverse matrix

This action is invariant under

i) general coordinate transformations (diffeomorphisms) $x^\mu \to x'^\mu(x)$

ii) local Lorentz rotations $\Psi(x) \to L(x)\Psi(x), \qquad L(x) \in SO(4) \simeq SU(2)_{\boldsymbol{L}} \times SU(2)_{\boldsymbol{R}}.$

$$\omega_\mu \to L^{-1} \omega_\mu L + L^{-1} \partial_\mu L$$

transforms as a Yang – Mills gauge field

Cartan's formulation of general relativity (early 1920's) uses precisely these variables:

Independent variables, instead of the metric tensor, are

1) *vierbein* or frame field e^A_μ, $g_{\mu\nu} = e^A_\mu e^A_\nu$, $A = 1, 2, 3, 4.$ 16 var's

2) spin connection $\omega^{AB}_\mu = -\omega^{BA}_\mu$ Yang – Mills potential of the Lorentz SO(4) group 24 var's

SO(4) Yang – Mills field strength or Cartan curvature:

$$F^{AB}_{\mu\nu} = \partial_\mu \omega^{AB}_\nu - \partial_\nu \omega^{AB}_\mu + \omega^{AC}_\mu \omega^{CB}_\nu - \omega^{AC}_\nu \omega^{CB}_\mu$$

Gravitation action:

$$S = \int d^4x \left(-\lambda^4 \det(e) - M_P^2 \frac{1}{4} \epsilon^{\kappa\lambda\mu\nu} \epsilon_{ABCD} F^{AB}_{\kappa\lambda} e^C_\mu e^D_\nu \right)$$

$$M_{\mathrm{P}} = \frac{1}{\sqrt{16\pi G_N}} = 1.72 \cdot 10^{18}\ \mathrm{GeV}$$

$$\lambda = 2.39 \cdot 10^{-3}\ \mathrm{eV}$$

Classically, and with no sources, it is equivalent to Einstein's theory based on Riemann geometry

Proof: The action in quadratic in ω_μ, so saddle point integration in ω_μ is exact.

Saddle-point equation for ω_μ:

$$D_\mu^{AB} e_\nu^B - D_\nu^{AB} e_\mu^B \equiv 2T_{\mu\nu}^A = 0, \qquad D_\mu^{AB} = \partial_\mu \delta^{AB} + \omega_\mu^{AB}$$

this combination is called **torsion**

$$\bar{\omega}_\mu \sim e^{-1}\partial_\mu e$$

24 algebraic equation on 24 components of ω_μ^{AB} determine the saddle-point uniquely as

$$\bar{\omega}_\mu^{AB}(e) = \frac{1}{2}e^{A\kappa}(\partial_\mu e_\kappa^B - \partial_\kappa e_\mu^B) - \frac{1}{2}e^{B\kappa}(\partial_\mu e_\kappa^A - \partial_\kappa e_\mu^A) - \frac{1}{2}e^{A\kappa}e^{B\lambda}e_\mu^C(\partial_\kappa e_\lambda^C - \partial_\lambda e_\kappa^C)$$

Substituting the saddle-point value back into the action, one recovers identically the Einstein – Hilbert action written in terms of $g_{\mu\nu}$:

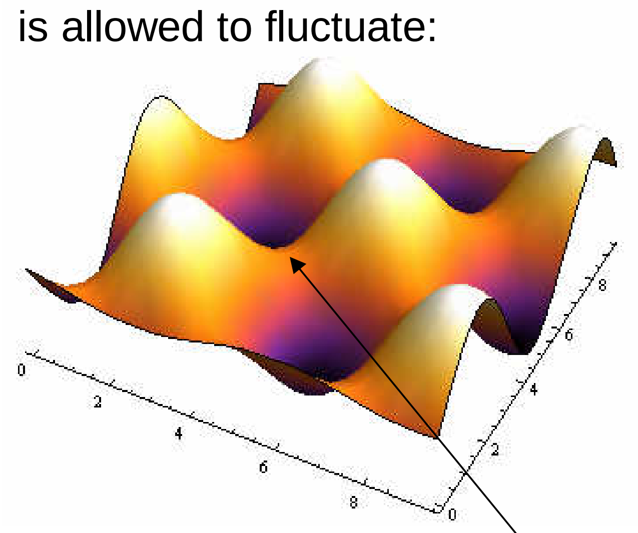
$$\left(-\frac{1}{4}\epsilon^{\kappa\lambda\mu\nu}\,\epsilon_{ABCD}\,F_{\kappa\lambda}^{AB}(\bar{\omega})e_\mu^C\,e_\nu^D\right) \quad = \quad \bar{R}\sqrt{g} \qquad \textbf{!}$$

Torsion appears to be zero dynamically, even if one allows it, as in Cartan formulation.

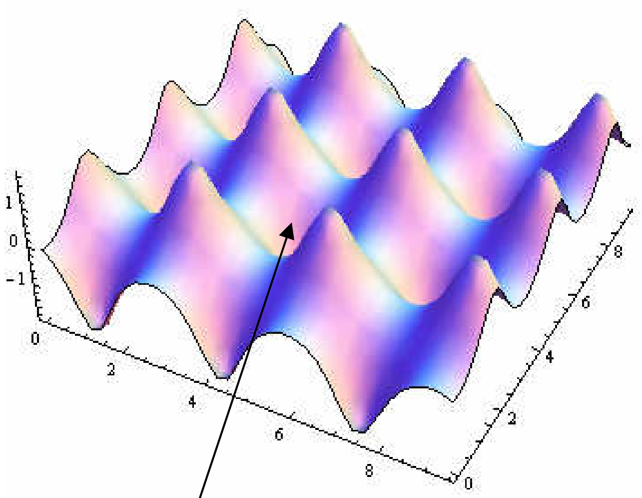
In the presence of fermion sources, torsion is nonzero and induces local 4-fermion Interaction. However, its effect is totally negligible, at least in the range of applicability of the derivative expansion [D.D., Tumanov and Vladimirov (2011)].

Sign Problem of quantum gravity

In quantum gravity, space-time is allowed to fluctuate:



curvature fluctuates, too, and can be locally of any sign:



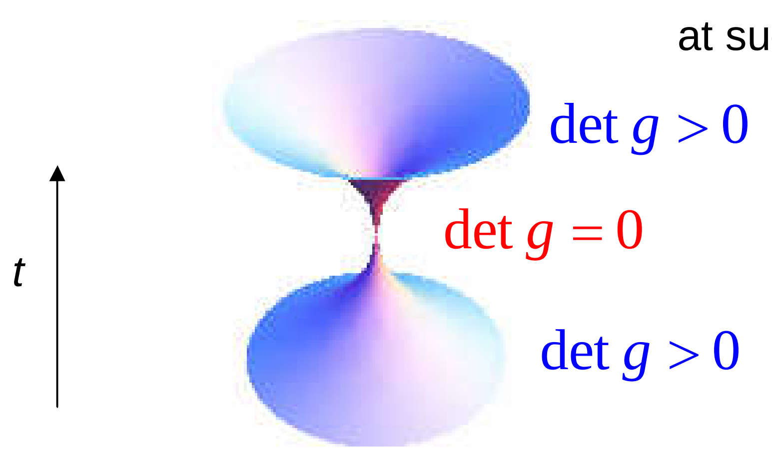
around saddle points curvature is negative, R < 0.
around maxima and minima curvature is positive.

The standard Einstein – Hilbert action of General Relativity $\int d^4x\, R\sqrt{\det g_{\mu\nu}}$ is not sign-definite! Therefore, it cannot restrict quantum fluctuations of the metrics!

What about $\int d^4x\, R^2\sqrt{\det g_{\mu\nu}} = \int d^4x \dfrac{\left(\epsilon^{ABCD}\,\epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda}^{\ AB}\, e_\mu^C e_\nu^D\right)^2}{\epsilon^{ABCD}\,\epsilon^{\kappa\lambda\mu\nu} e_\kappa^A e_\lambda^B e_\mu^C e_\nu^D}$?

What about the cosmological term $\int d^4x\sqrt{\det g_{\mu\nu}} = \dfrac{1}{4!}\int d^4x\, \epsilon^{ABCD}\,\epsilon^{\kappa\lambda\mu\nu}\, e_\kappa^A e_\lambda^B e_\mu^C e_\nu^D$?

If metrics is allowed to fluctuate arbitrarily, $\det g_{\mu\nu}(x, y, z, t)$ can become zero at some point *t*:



at such point the space effectively looses one dimension

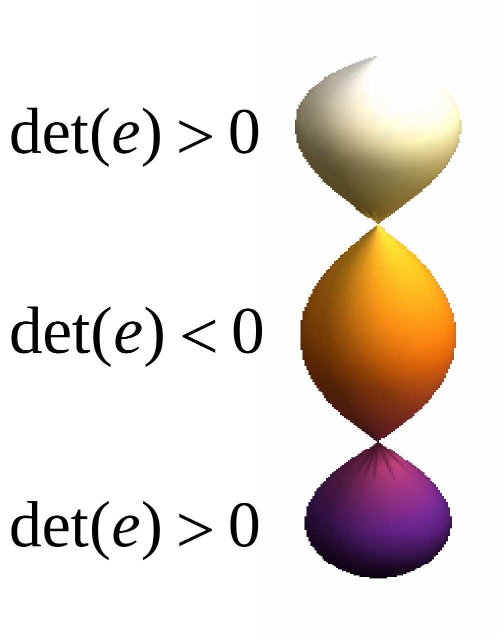
$$g_{\mu\nu} = e_\mu^A e_\nu^A, \quad A = 1, 2, 3, 4$$

$$\mu, \nu = 1, 2, 3, 4$$

metric tensor

vierbein (frame, rep'ere, tetrad)

$$g \equiv \det(g_{\mu\nu}) = \det(e_\mu^A e_\nu^A) = \det(e)\det(e^T) = (\det(e))^2$$

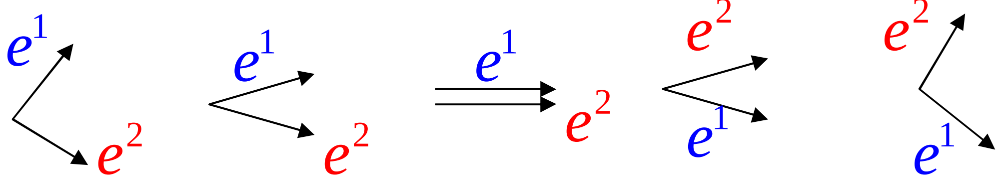


If det(e) passes through zero as ***det(e) ~ t***, then det(g) passes through zero as $\det(g) \sim t^2$

$\sqrt{\det(g)}$ should be then understood as

$\sqrt{\det(g)} \sim \det(e) \sim t$ changing sign, and not as modulus

$$\sqrt{\det(g)} \neq |t|$$

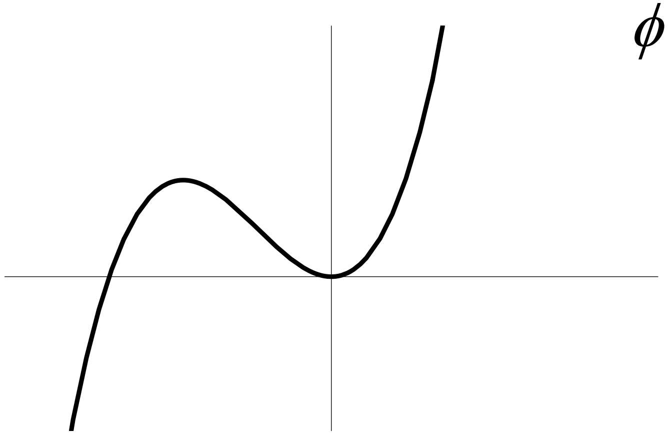


Passing through zero, det(e) changes sign by continuity

Therefore, the possible action term $\int R^2 \sqrt{g} = \int R^2 \det(e)$ is, strictly speaking, not sign-definite, too.

In Euclidian space-time we write quantum amplitude as ***exp(- Action) .***
(used in thermodynamics, in tunneling problems, *etc.)*
If the action is not positive-definite, there is no ground state!

In Minkowski space-time we write the amplitude as ***exp(i Action) .***
(used for real-time problems*.)*
If the action is not sign-definite, there will be problems in defining Feynman propagator for gravitons in arbitrary curved space!



ϕ^3 theory does not restrict large quantum fluctuations, even though perturbation theory may be well defined.

Minkowski space-time with e^{iS} doesn't seem to help, if the action can have any sign, and is unbounded: one cannot define Feynman's propagator, and there is tunneling to a bottomless state!

General covariance is a "curse" that makes any diffeomorphism-invariant action bottomless!

The Sign Problem of quantum gravity:
Large fluctuations of the frame and/or of spin connection are not restricted!

How, then, to define the path integral for Quantum Gravity? Use in part
fermionic anticommuting variables instead of bosonic ones! [DD, arXiv:1109.0091]

Integration over anticommuting, called Grassmann, variables has been introduced by F. Berezin (1965):

$$\psi_i \psi_j = -\psi_j \psi_i, \quad \psi_i \psi_j^\dagger = -\psi_j^\dagger \psi_i \quad \psi_i^\dagger \psi_j^\dagger = -\psi_j^\dagger \psi^\dagger{}_i$$

$$\int d\psi = 0, \qquad \int d\psi\, \psi = 1 \qquad \int d\psi^\dagger = 0, \qquad \int d\psi^\dagger\, \psi^\dagger = 1$$

Berezin integrals are well defined for whatever sign of the (multi) fermion action:

$$\int d\psi^\dagger\, d\psi \exp\left(\psi_i^\dagger\, A_{ij}\, \psi_j\right) = \epsilon^{i_1 \ldots i_N}\, \epsilon^{j_1 \ldots j_N}\, A_{i_1 j_1} \ldots A_{i_N j_N} = \det(A)$$

$$\int d\psi^\dagger\, d\psi \exp\left(\psi_i^\dagger \psi_j^\dagger\, A_{ij,kl}\, \psi_k \psi_l\right) = \epsilon^{i_1 \ldots i_N}\, \epsilon^{j_1 \ldots j_N}\, A_{i_1 i_2,\, j_1 j_2} \ldots A_{i_{N-1} i_N,\, j_{N-1} j_N} \qquad \textit{etc.}$$

The idea is to present the frame field as a **composite spinor** bilinear combination:

vierbein $$\hat{e}^{A}_{\mu} = \frac{1}{2}\psi^{\dagger}\gamma_{A}\nabla_{\mu}\psi - \frac{1}{2}(\nabla_{\mu}\psi)^{\dagger}\gamma_{A}\psi$$

$$\nabla_{\mu} = \partial_{\mu} + \frac{1}{8}\omega^{AB}_{\mu}[\gamma_{A}\gamma_{B}]$$

spin connection, gauge field

transforms correctly!

$$\hat{g}_{\mu\nu} = \hat{e}^{A}_{\mu}\hat{e}^{A}_{\nu}$$

metric tensor

History of composite frames:

- K. Akama (1978)
- G. Volovik (1990) [superfluid $^{3}\boldsymbol{He}-B$]
- C. Wetterich (2005, 2011)

use *ordinary* derivatives, not covariant ⟹ e^{A}_{μ} is not a Lorentz vector as it should be !

Standard Dirac action in *d*-dim curved space

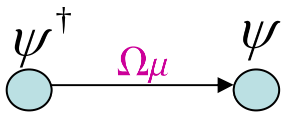
$$S = \int d^{d}x\, \epsilon^{\mu_1\ldots\mu_d}\, \epsilon^{A_1\ldots A_d}\, e^{A_1}_{\mu_1}\ldots e^{A_{d-1}}_{\mu_{d-1}}\left(\psi^{\dagger}\gamma_{A_d}(\nabla_{\mu_d}\psi) - (\nabla_{\mu_d}\psi)^{\dagger}\gamma_{A_d}\psi\right)$$

is in fact the cosmological term in disguise:

$$S = \int d^{d}x\, \epsilon^{\mu_1\ldots\mu_d}\, \epsilon^{A_1\ldots A_d}\, e^{A_1}_{\mu_1}\ldots e^{A_{d-1}}_{\mu_{d-1}} e^{A_d}_{\mu_d} = \int d^{d}x\, \det(e) = \int d^{d}x\sqrt{g}$$

All such kind of actions can be easily UV **regularized** by putting them on a lattice.

Discretized frame field:



parallel transporter

$$\hat{e}^A_\mu = \frac{1}{2}\psi^\dagger\gamma_A\nabla_\mu\psi - \frac{1}{2}(\nabla_\mu\psi)^\dagger\gamma_A\psi$$

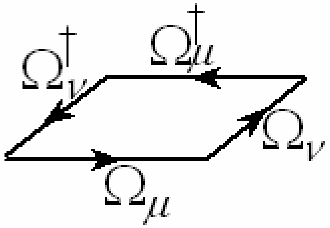
$$\to \frac{1}{2a}[\psi^\dagger(x)\gamma_A\,\Omega_\mu(x+a/2)\psi(x+a)-\psi^\dagger(x+a)\Omega^\dagger_\mu(x+a/2)\gamma_A\psi(x)]$$

Discretized connection = unitary SU(2) x SU(2) matrices living on lattice links:

$$\omega^{AB}_\mu \quad \to \quad \Omega_\mu = \exp\left(a\,\omega^{AB}_\mu[\gamma_A\gamma_B]/8\right), \qquad \Omega^\dagger{}_\mu = \exp\left(-a\,\omega^{AB}_\mu[\gamma_A\gamma_B]/8\right)$$

Discretized curvature:

$$F^{AB}_{\mu\nu} \quad \to \quad \Omega_{\mu\nu} =$$

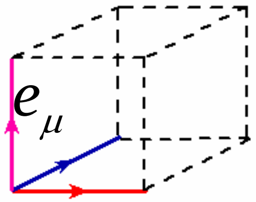


$$= 1_{4\times 4} + \frac{a^2}{8}F^{AB}_{\mu\nu}[\gamma_A\gamma_B] + O(a^3)$$

plaquette

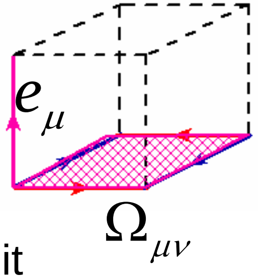
in any number of dimensions *d*

Discretized `cosmological term' action: $$\sum_x \epsilon^{\mu_1\cdots\mu_d}\, Tr(e_{\mu_1}\ldots e_{\mu_d})$$



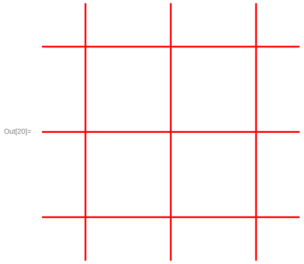
gauge- and diffeomorphism-invariant!

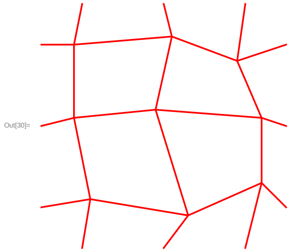
Discretized `Einstein – Hilbert' action: $$\sum_x \epsilon^{\mu_1\cdots\mu_d}\, Tr(e_{\mu_1}\ldots e_{\mu_{d-2}}\Omega_{\mu_{d-1}\mu_d})$$



Such actions define the same general covariant theory in the continuum limit

for rectangular and arbitrarily distorted lattices:





in fact, starting from d=3
one has to use **simplices:**
triangles, tetrahedra, *etc*.

Cf. two lattice gauge theories

$$\prod_{links} dU \exp\left(\beta\sum_x TrU_{plaquette}\right) \to \int DA_\mu \exp(-\frac{1}{2g^2}\int d^4x\, Tr(F_{\mu\nu}F_{\mu\nu}))$$

diffeomorphism
-noninvariant

$$\prod_{links} dU \exp\left(\beta\sum_x TrU_{12}U_{34}\right) \to \int DA_\mu \exp(-\frac{1}{2g^2}\int d^4x\, \epsilon^{\kappa\lambda\mu\nu} Tr(F_{\kappa\lambda}F_{\mu\nu}))$$

diffeomorphism
-invariant

Regularized partition function for quantum gravity:

connection

8 spinor fields

4 spinor fields

$$Z = \prod_{links} \int d\Omega_{\mu} \prod_{sites} \int \overbrace{d\psi^{\dagger}\, d\psi}^{\text{frame}} \; \exp(\lambda_1 S_{cosm} + \lambda_2 S_{EH} + \ldots)$$

Haar measure SU(2) x SU(2)
normalized to unity

dimensionless
“coupling constants”

D.D. 1109.0091
C. Wetterich
1110.1539

The theory is well-defined, well-behaved in the ultraviolet,
explicitly gauge invariant under local Lorentz group,
and diffeomorphism-invariant in the continuum limit !

The lattice does not need to be regular: it can be arbitrarily deformed.

This is a very **unusual lattice field theory** – with many-fermion vertices
but no bilinear term for the fermion propagator!

Nevertheless, fermions “propagate” since vertices contain spinors belonging
to neighbour lattice sites.

How to work with such new kind of lattice gauge theory?

At each lattice one integrates over 8 Grassmann variables $\int d\psi_1^{\dagger} d\psi_2^{\dagger} d\psi_3^{\dagger} d\psi_4^{\dagger} d\psi_1 d\psi_2 d\psi_3 d\psi_4$

The action has also 8 operators $S(x) \sim \psi^{\dagger}\psi^{\dagger}\psi^{\dagger}\psi^{\dagger}\psi\psi\psi\psi$. One has to Taylor-expand $e^{\sum_x S(x)}$

such that there are precisely 8 fermion operator per site, otherwise the integral = 0.

After integrating over link variables using $\int dU\, U^{\dagger\alpha}{}_{\beta} U^{\gamma}_{\delta} = \frac{1}{2} \delta^{\alpha}_{\delta}\, \delta^{\gamma}_{\beta},$ $\quad \int dU\, U^{\alpha}_{\beta} U^{\gamma}_{\delta} = \frac{1}{2} \epsilon^{\alpha\gamma}\, \epsilon_{\beta\delta}$

one gets only gauge invariant combinations of fermion operators $(\psi^{\dagger}\psi), \quad (\psi^{\dagger}\epsilon\psi^{\dagger}), \quad (\psi\epsilon\psi)$

The partition function is in fact a sum over all types of closed loops, closed surfaces and closed 3-volumes!

Numerical simulations are possible: one can generate closed loops by *e.g.* Metropolis-like procedure.

Toy model: 2d quantum gravity

In 2d the partition function can be computed exactly by summing over *all* closed loops, which is a good way to test approximate numerical methods, to be used in d>2.

Some exact results:

number of points
on the lattice

physical
volume

$$< \int d^2x \det(e) > \quad = \quad \frac{N}{\lambda_1}$$

$$< V >$$

extensive quantity, good!

fermions are non-compressable!

$$< \int d^2x \det(e) R > \quad = \quad 0$$

space is on the average flat, good!

average torsion is also zero

physical
volume
susceptibility

$$< (\int d^2x \det(e))^2 > - < \int d^2x \det(e) >^2 \quad = \quad -\frac{N}{\lambda_1^2}$$

$$< V^2 > - < V >^2$$

it's **quantum**
gravity, not
classical !

this is also nice!

A typical difficulty in other models of discretized gravity: when allowed to fluctuate nonperturbatively, the space gets either crunched or forms chaotic `branched polymers'.

Here the fluctuating space is smooth, because fermions are non-compressible !

Spinor quantum gravity is a very rich and yet unexplored theory. It turns out that, depending on the values of dimensional coupling constants, there can be several phases!

Two continuous symmetries that can be spontaneously broken:

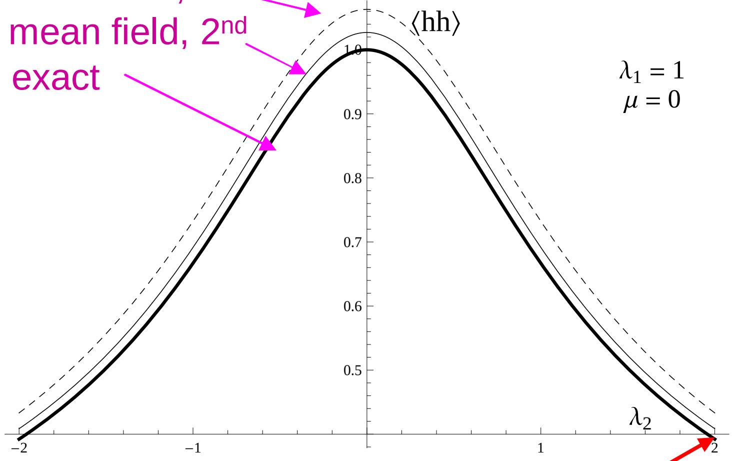
1) Chiral symmetry, $\psi \to e^{i\alpha\gamma_5}\psi, \quad \psi^\dagger \to \psi^\dagger e^{i\alpha\gamma_5};$

2) Fermion number conservation: $\psi \to e^{i\alpha}\psi, \quad \psi^\dagger \to \psi^\dagger e^{-i\alpha};$

$$e^A_\mu \to e^A_\mu$$

The phase diagram can be found by a relatively simple **mean field** method.
It works quite accurately, as can be checked by comparison with an exactly solvable model:

chiral condensate



mean field, 1st

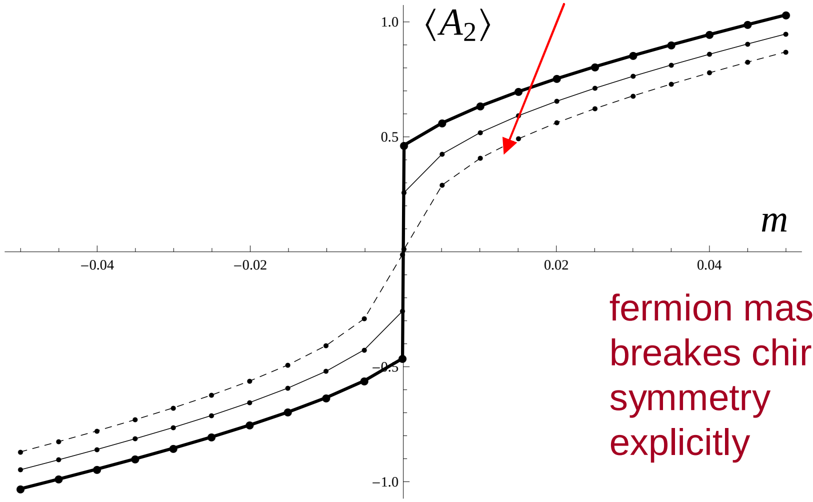
mean field, 2nd

exact

phase transition is II order

chiral condensate

phase transition point



fermion mass, breakes chiral symmetry explicitly

When symmetry is broken the order parameter is non-analytical

We check that in the broken phase the “effective chiral Lagrangian” for long-range Goldstone fields is explicitly diffeomorphism-invariant:

$$L = \int dx < \sqrt{g}\; g^{\mu\nu} > \;\; \partial_{\mu}\alpha\, \partial_{\nu}\alpha$$

We need< however, *__all__* degrees of freedom to be long-ranged, not only the Goldstone mode. To that end, one needs to stay at exactly the phase transition point.

We expect that the low-energy Einstein action will come out automatically there, since diffeomorphism-invariance is supported by construction.

To see it, one can introduce the **effective action** by means of a Legendre transform,

stress-energy source

$$\exp(-W[\Theta^{\mu\nu}]) = \prod_{links} \int d\Omega_{\mu} \prod_{sites} \int d\psi^{\dagger}\, d\psi \;\; \exp(S + \int \Theta^{\mu\nu}\, \hat{g}_{\mu\nu})$$

generating functional

4-fermion operator

$$g^{class}_{\mu\nu} = \frac{\delta W}{\delta \Theta^{\mu\nu}} \quad \Rightarrow \quad \Theta^{\mu\nu}(g^{class}_{\mu\nu}) \quad \Rightarrow \quad \Gamma[g^{class}_{\mu\nu}] = W[\Theta^{\mu\nu}] - \Theta^{\mu\nu}\, g^{class}_{\mu\nu}$$

classical metrics

effective action, must be $\int R(g^{class})\sqrt{g^{class}}$ **!!**

Speculations: Unifying quantum gravity with the Standard Model ?

The Standard Model is based on the $SU(3)_c \times SU(2)_W \times U(1)$ gauge group, and has 64 real fermion dof's per generation.

With a composite frame field built as a bilinear spinor current, the content of QG are also fermions and the gauge field of the local Lorentz group SU(2) x SU(2).

In the SM quantum fluctuations are tamed, and in the QG the fluctuations are now also tamed. Why not unify them?!

We want the spinor fields to carry exactly the same number of dof's as the frame field, equal to *d x d*. This doesn't happen in any number of space-time dimensions.

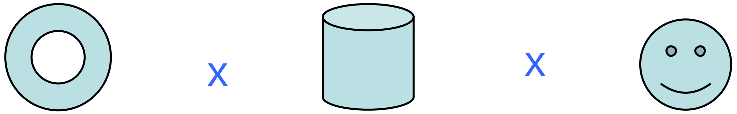
The dimension of the two spinor representations of the SO(d=2n) group is $2^{\frac{d}{2}} \overset{!}{=} d^2$.
This equation has only one solution: ***d=16***.

The 256-dimensional spinor representation of SO(16) falls neatly into ***four*** generations of the Standard Model.

One needs a mechanism to break spontaneously the SO(16) rotational gauge group.

The action in d=16 may have 7 terms: $\underset{16}{(e \wedge e \ldots \wedge e)}, \quad \underset{14}{(e \wedge e \ldots \wedge e \wedge F)}, \ldots, \underset{2}{(e \wedge e \wedge F \ldots \wedge F \wedge F)}$
with *a priori* arbitrary coefficients.
Write them in terms of fermions.

It may happen that the **fermion condensates** $< \psi_i^\dagger \psi^j > \quad \neq \quad 0$ break spontaneously the rotational SO(16) symmetry, for example, by compactifying the 16d space down to *e.g.* a direct product of several low-dimensional spheres, or whatever



and breaks the SO(16) gauge group down to the gauge group of the Standard Model *and* Lorentz gauge group: [Coleman – Mandula theorem works for flat space only!]

$$SO(16) \supset \underset{\text{gravity spin connection}}{(SU(2)_L \times SU(2)_R)} \times \underset{\text{Standard Model}}{SU(3)_C \times SU(2)_W \times U(1)_Y}$$

256 fermion fields needed to describe the 16d metric, fits precisely 4 generations of the SM.

Conclusions

1. ***All* general covariant action terms are not sign-definite**. It prevents from defining a quantum theory where large fluctuations are allowed.
2. In order to define a quantum theory properly, one presents the frame field as a composite bilinear fermion “current”. This will be then **spinor quantum gravity**.
3. It is easily regularized at short distances by imposing a diffeomorphism-preserving lattice. Fermion path integrals are **well defined and well-behaved**.
4. It is an exciting new kind of theory, with potentially rich phase structure associated with spontaneous breaking of continuous symmetries by **fermion condensates**.
5. **Einstein’s theory is expected in the low-energy limit** at the phase transition point(s) where the original lattice structure is “forgotten”.
6. Spinor quantum gravity and the Stadard Model share the same basic degrees of freedom, *viz.* **fermion and gauge fields.** Therefore new ways arise to unify the two.

Conceptual problem

Supposing one has a well-defined quantum gravity at hand, how to check it has the correct infrared limit – the Einstein's gravity ?

In general, one has to compute diffeomorphism-invariant correlation functions, like

$$I_1(s) = \frac{< \int dx\sqrt{g} \int dy\sqrt{g}\ \delta(S(x,y) - s) >}{< \int dz\sqrt{g} >} \left(= 2\pi^2 s^3\right)$$

interval over geodesic

$$S(x,y) = \int dt \sqrt{g_{\mu\nu}\ \dot{x}^{\mu}(t)\dot{x}^{\nu}(t)}$$

$$I_2(s) = \frac{< \int dx\sqrt{g} R(x) \int dy\sqrt{g} R(y) \delta(S(x,y) - s) >}{< \int dz\sqrt{g} >} \left(\sim \frac{1}{s^3}\right)$$

Newton's law in disguise

In our case, however, $g_{\mu\nu}$ is a fermion operator and cannot be used.
One can introduce the classical metric tensor by means of a Legendre transform: