Shining Black Holes:
Classes of Plasma and Field Configurations and Magnetic Field Generation Processes *

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Conventional currentless disks that are commonly assumed to surround black holes are shown to be subject to the excitation of magneto gravitational modes [1] associated with the gradients of the rotation frequency, of the plasma density and temperature combined with the effects of gravity. Thus stationary current carrying plasma configurations that can be found by a fully non-linear analysis have been looked for and their features connected to those of the magneto-gravitational modes found by a linearized analysis.

In particular, two classes of plasma and field axisymmetric configurations are found all of them involving periodic sequences of plasma rings or solitary rings. These are: i) Localized Differential Rotator configurations [2] that are connected mainly to the radial gradient of the rotation frequency; ii) Localized Rigid Rotor configurations that are connected to the product of the vertical component of the gravitational force and the radial density gradient. The latter class of configurations does not require, unlike the former [2], the presence of a seed magnetic field.

Analytical representations of both classes of configurations are derived and the compatibility of these configurations with the presence of accretion processes is pointed out.

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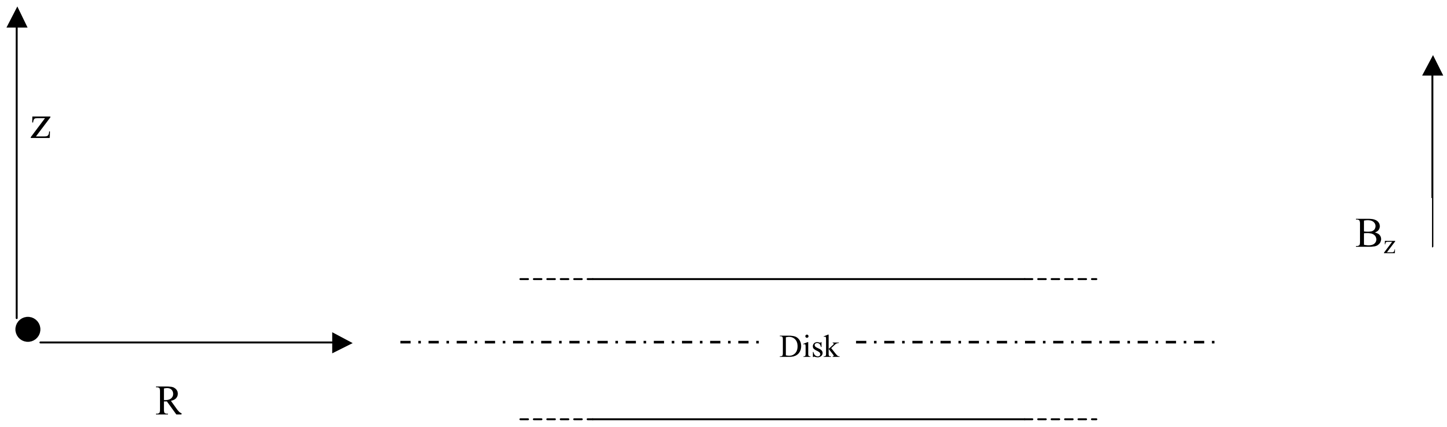
[1] B. Coppi, *A&A* **504**, 321 (2009)

[2] B. Coppi, *Phys. Plasmas* **18**, 032901 (2011)

Currentless Plane Disk Model



$$0 = -\frac{d}{dz}\left(p_i + p_e\right) - \rho z \Omega_k^2$$

Ω_k = Keplerian Frequency $\qquad \Omega_k^2 = G\dfrac{M_*}{R^3}$

$p = p_e + p_i$

Relevant Driving Factors for Different (from the Conventional Disk) Configurations

$$\mathbf{D} = \nabla \times \left(\rho \nabla \Phi_G + \rho \Omega^2 R \mathbf{e}_R \right) = -\nabla \Phi_G \times \nabla \rho + \nabla \times \left(\rho \Omega^2 R \mathbf{e}_R \right)$$

$$= -\frac{\partial \phi_G}{\partial z} \frac{\partial \rho}{\partial R} \mathbf{e}_\phi + \frac{\partial \phi_G}{\partial R} \frac{\partial \rho}{\partial z} \mathbf{e}_\phi + \frac{\partial}{\partial z} \left(\rho \Omega^2 R \right) \mathbf{e}_\phi$$

Therefore

$$D = R \frac{\partial}{\partial z} \left(\rho \Omega^2 \right) + \frac{\partial \phi_G}{\partial R} \frac{\partial \rho}{\partial z} - \frac{\partial \phi_G}{\partial R} \frac{\partial \rho}{\partial R}$$

$$= \left(\frac{\partial \rho}{\partial z} \right) \left[R \Omega^2 + \frac{\partial \phi_G}{\partial R} \right] + R \rho \frac{\partial}{\partial z} \Omega^2 - \frac{\partial \phi_G}{\partial z} \frac{\partial \rho}{\partial R}$$

Note: A non-isotropic electron distribution in velocity space, for instance with

$$T_{e\perp} \neq T_{e\|},$$

can drive the growth of the magnetic fields toward significant values starting from a very small initial perturbation of the electron distribution. Can gravity combined with plasma inhomogeneities continue the process of magnetic field generation after the anisotropy has vanished?

Magneto Gravitational Modes that are linearly unstable have been found (Coppi, *A&A*, **504**, 2009) which include both

$$\Omega_0 z \frac{\partial}{\partial R}\rho \quad \text{and} \quad 2\Omega_0 R \frac{\partial}{\partial z}\left[\rho\left(\delta\Omega\right)\right]$$

as relevant driving factors. A seed magnetic field B_z^0 is required for the existence of these modes.

Driving Factors for Magneto Gravitational Modes

$$z\Omega_K^2 \frac{\partial \hat{\rho}}{\partial R} + 2\Omega_K \frac{\partial}{\partial z}\left(\rho \hat{\Omega} R\right)$$

Gravitational + Inertia terms (perturbed) entering final dispersion equation

$$-\Omega_K^2 \left(\frac{\partial}{\partial k}\hat{\rho}\right) k_R + 2\Omega_K (ik)\left(\rho \hat{\Omega}_k R\right)$$

Fourier Transform

$$\hat{\rho} = \tilde{\rho}(z)\exp\left[-ik_R\left(R - R_0\right)\right]$$

$$\hat{\rho}_k = \tilde{\rho}_k(k)\exp\left[-ik_R\left(R - R_0\right)\right]$$

$$ik_R \tilde{\xi}_{Rk} \simeq ik\tilde{\xi}_{zk}$$

$\tilde{\xi} =$ displacement amplitude

$$\hat{\Omega} \simeq -\frac{d\Omega}{dR}\hat{\xi}_R$$

$$\hat{\rho} = -\hat{\xi}_z \frac{d\rho}{dz} - \rho \nabla \cdot \hat{\xi}$$

Dispersion Equation and Marginal Stability Condition

$$0 \simeq (kH_0)^2 \left(\varepsilon - \frac{k^2}{k_0^2} \right) \tilde{\xi}_{zk} + k \frac{d^2}{dk^2} \left(k \tilde{\xi}_{zk} \right) + \Lambda_T k_R^2 \frac{d^2}{dk^2} \tilde{\xi}_{zk}$$

$$\Lambda_T \equiv \frac{6}{5} \left[\frac{d \ln T}{dz} \Big/ \left(\frac{d \ln \rho}{dz} \right) - \frac{2}{3} \right]$$

$$\varepsilon \equiv \frac{1}{\Omega_K^2} \left(\Omega_D^2 k_R^2 \mathrm{v}_{Az}^2 \right) \qquad\qquad H_0 \equiv -\left(d \ln \rho / dz \right)^{-1}$$

$$k_0^2 \equiv \frac{\Omega_K^2}{\mathrm{v}_{Az}} \qquad \Omega_D^2 \equiv -2\Omega_K \frac{d\Omega_K}{dR} R = 3\Omega_K^2 \qquad k_R^2 \simeq 3k_0^2 .$$

The lowest eigensolution $\tilde{\xi}_{zk} = \tilde{\xi}_{zk}^0 \exp\left(-\Delta_z^2 \, k^2 / 2 \right)$ corresponds to $\Lambda_T = 0$.

Dispersion Equation for Unstable Modes

$$\hat{\rho} = \tilde{\rho}(z)\exp\left[\gamma_0 t - ik_R\left(R - R_0\right)\right]$$

$$0 \simeq \left\{\frac{7}{3}\left(\frac{\gamma_0}{\mathrm{v}_A}\right)^2 + \left(\varepsilon k^2 - \frac{k^4}{k_0^2}\right)\right\}\tilde{\xi}_{zk} + \frac{1}{H_0^2}\left\{k\frac{d}{dk^2}\left(k\tilde{\xi}_{zk}\right) + \Lambda_T k_R^2 \tilde{\xi}_{zk}\right\}$$

$$\tilde{\xi}_{zk} \propto \exp\left(-\Delta_z^2 \frac{k^2}{2}\right)$$

$$\frac{7}{3}\left(\frac{\gamma_0}{\mathrm{v}_A}\right)^2 \simeq \left(\Delta_z k_R\right)^2 \frac{\Lambda_T}{H_0^2} = 3\frac{k_0}{H_0}\Lambda_T$$

$$\Delta_z^2 = \frac{H_0}{k_0} \qquad \gamma_0 \simeq 3\left(\frac{\Omega_k \mathrm{v}_A}{7H_0}\Lambda_T\right)^{1/2}$$

$$\varepsilon = 3\left(\frac{\Delta_z^2}{H_0^2}\Lambda_T\right)$$

$$\Lambda_T < 1$$

Next (odd) Eigenfunction

$$\tilde{\xi}_{zk} \propto \exp\left(-\frac{\Delta_z^2}{2}k^2\right)$$ odd

$$\tilde{\xi}_{zk} \propto k^2 \exp\left(-\frac{\Delta_z^2}{2}k^2\right)$$ even

$$\tilde{\xi}_{zk} \propto \exp\left(\frac{z^2}{\Delta_z^2}-1\right)\exp\left(-\frac{z^2}{2\Delta_z^2}\right)$$

$$\gamma_0^2 \simeq \frac{6}{7}\frac{\mathrm{v}_A}{H_0}\left[\frac{\sqrt{33}}{5}\left(\eta_T-\frac{2}{3}\right)\Omega_k+\frac{\mathrm{v}_A}{H_0}\right]$$

↑ M.G.I. (magneto-gravitational-instability)

↑ M.R.I. (magneto-rotational-instability)

Quadratic Form

$$\Omega_k^2\left[\left\langle k^2\tilde{\xi}_{zk}^2\right\rangle - \frac{\Lambda_T}{H_0^2}\left\langle\left(\frac{d}{dk}\tilde{\xi}_{zk}\right)^2\right\rangle\right] = \mathrm{v}_A^2\left\langle k^4\tilde{\xi}_{zk}^2\right\rangle + \frac{\Omega_k^2}{H_0^2}\left\langle\frac{d}{dk}\left(k\tilde{\xi}_{zk}\right)^2\right\rangle$$

$$\langle\ \rangle \equiv H_0\int_{-\infty}^{+\infty} dk$$

Sources of instability

$$\frac{d\Omega}{dR} \quad \text{and} \quad \frac{dT}{dz}$$

to be associated with the sources of magnetic field configurations emerging from the relevant non-linear analysis.

Basic Assumptions for Non-linear Configurations

Given the intent to identify the simplest plasma and field configurations that can (theoretically) exist around compact collapsed objects such as black holes we limit our analysis to axisymmetric geometries with the following points:

- a) perfectly conducting plasma regimes are considered and, consequently,

$$\mathbf{V}=\alpha_{\mathrm{v}}\mathbf{B}+\Omega\left(\psi\right)R\mathbf{e}_{\varphi}$$

  where $\mathbf{V}$ is the plasma flow velocity, $\psi=\psi\left(R,z\right)$ is the magnetic surface function and we use cylindrical coordinates.
- b) no appreciable poloidal flow velocity is included.
- c) the relevant particle distributions in velocity space are close to Maxwellian and referring to a scalar pressure ($\boldsymbol{P}=p\boldsymbol{I}$) is appropriate in this case

Basic Assumptions for Non-linear Configurations (cont.)

- d) at first we assume that the relevant Lorentz force does not have a toroidal component and the relevant magnetic field configurations are represented by

$$\mathbf{B} = \frac{1}{R}\left[\nabla\psi \times \mathbf{e}_\phi + I\left(\psi\right)\mathbf{e}_\phi\right] \ .$$

  In this case the Lorenz force $\mathbf{F}_L$ is given simply by

$$\mathbf{F}_L = \frac{1}{4\pi R^2}\left(\Delta_*\psi + I\frac{dI}{d\psi}\right)\nabla\psi$$

  where

$$\Delta_*\psi \equiv \frac{\partial^2\psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\psi\right).$$

Basic Assumptions for Non-linear Configurations (cont.)

- e) a Newtonian gravitational potential Φ_G is included for simplicity. In particular, for the relative thin structures that we shall analyze

$$\nabla \Phi_G \simeq -\frac{V_K^2}{R}\left(\mathbf{e}_R + \frac{z}{R}\mathbf{e}_z\right)$$

where

$$V_K^2 \equiv \frac{GM_*}{R} \equiv \Omega_K^2 R^2$$

and Ω_K is the Keplerian frequency. We observe that when considering scale distances which are relatively close to black holes we have found it convenient to make use of effective gravitational potentials in order to include relevant General Relativity effects in the theory (Coppi, 2011).

Relevant Form of the Master Equation

Referring to the total momentum conservation

$$-\rho\left(\nabla\Phi_G + \Omega^2 R\,\mathbf{e}_R\right) = -\nabla p + \frac{1}{c}\mathbf{J}\times\mathbf{B} \tag{1}$$

we observe that

$$\begin{aligned}\nabla\times\left(\rho\nabla\Phi_G + \rho\Omega^2 R\,\mathbf{e}_R\right) &= \mathbf{e}_\phi\left\{\frac{\partial\rho}{\partial z}\left(R\Omega^2 + \frac{\partial\Phi_G}{\partial R}\right)\right. \\ &\quad \left. + \rho R 2\Omega\frac{\partial\Omega}{\partial z} - \frac{\partial\rho}{\partial R}\frac{\partial\Phi_G}{\partial z}\right\}\end{aligned} \tag{2}$$

and

$$\begin{aligned}\nabla\times\left(\frac{1}{c}\mathbf{J}\times\mathbf{B}\right) = \frac{1}{4\pi}\nabla\times\left(\mathbf{B}\cdot\nabla\mathbf{B}\right) &= \frac{1}{4\pi R^2}\left[-\frac{2}{R}\left(\Delta_*\psi + I\frac{dI}{d\psi}\right)\mathbf{e}_R\right. \\ &\quad \left. + \nabla\left(\Delta_*\psi\right)\right]\times\nabla\psi.\end{aligned} \tag{3}$$

Relevant Form of the Master Equation (cont.)

Therefore, we obtain the "Master Equation" (Coppi, 2011) that relates ψ to ρ

$$R\frac{\partial}{\partial z}\left(\Omega^2\rho\right)+\frac{\partial\rho}{\partial z}\frac{\partial\Phi_G}{\partial R}-\frac{\partial\rho}{\partial R}\frac{\partial\Phi_G}{\partial z} \tag{4}$$

$$\simeq \frac{1}{4\pi R^2}\left\{\left[\frac{2}{R}\left(\Delta_*\psi+I\frac{dI}{d\psi}\right)-\frac{\partial}{\partial R}\left(\Delta_*\psi\right)\right]\frac{\partial\psi}{\partial z}+\left[\frac{\partial}{\partial z}\left(\Delta_*\psi\right)\right]\frac{\partial\psi}{\partial R}\right\}.$$

We consider local plasma and field structures in an interval $|R-R_0|<R_0$ around $R-R_0$ and we indicate the characteristic scale distance over which $R-R_0$ and z vary by Δ_R and Δ_z, respectively. Here $\rho=\rho\left(R_*,\bar{z}^2\right)$, $\psi=\psi\left(R_*,\bar{z}^2\right)$ for

$$R_*\equiv\frac{R-R_0}{\Delta_R},\ \bar{z}\equiv\frac{z}{\Delta_z}\ \text{and}\ \Delta_R^2\lesssim\Delta\ll R_0^2.$$

In particular, we consider ρ to be an even function of both R_* and $\bar{z}$ that is positive for all values of R_* and $\bar{z}$.

Locally Rigid Rotor Configurations

The Locally Rigid Rotor configurations we consider are localized over a radial scale distances $|\Delta R| < R_0$. In particular, for these configurations

$$V_\phi = \Omega_0 R \quad \text{where} \quad \Omega_0 \equiv \Omega_k \left(R = R_0 \right).$$

Then

$$\Omega_0 R = \alpha_v B_\phi + \Omega(\psi) R \tag{1}$$

where $\nabla \cdot (\rho \mathbf{V}) = 0$ implies that $\alpha_v \rho = G(\psi)$ and

$$B_\phi = \left[\Omega_0 - \Omega(\psi) \right] \frac{R\rho}{G(\psi)}. \tag{2}$$

We note that in this class of configurations a seed magnetic field is not required.

Clearly, the simplest case to analyze is that with $B_\phi = 0$ and $\Omega(\psi) = \Omega_0$.

Locally Rigid Rotor Configurations

Then the Master Equation reduces to

$$\Omega_0^2\left[z\frac{\partial\rho}{\partial R}+3\left(R-R_0\right)\frac{\partial\rho}{\partial z}\right]$$

$$\simeq\frac{1}{4\pi R_0^2}\left\{\left[\left(\frac{\partial^3}{\partial z^3}\psi\right)\frac{\partial\psi}{\partial R}-\left(\frac{\partial^3}{\partial R^3}\psi\right)\frac{\partial\psi}{\partial z}\right]+\left[\left(\frac{\partial}{\partial z}\frac{\partial^2}{\partial R^2}\psi\right)\frac{\partial\psi}{\partial R}-\left(\frac{\partial}{\partial R}\frac{\partial^2}{\partial z^2}\psi\right)\frac{\partial\psi}{\partial R}\right]\right\}$$

and we note that in this case ψ can be either an odd or an even function of R_* and an even or odd function of z.

Note that the formation of the relevant magnetic configuration is connected to the term $F_z\,\partial\rho/\partial R$, that is, to the vertical component of the vertical force combined with the radial density gradient.

$$\mathrm{v}_{Az}^2\sim\mathrm{v}_{thi}^2\frac{\Delta_R}{\Delta_z}$$

Master Equation for Local Rigid Rotor Configurations

Referring to the relevant form of the Master Equation and considering the dependence of ψ on z indicated by

$$\psi_1 = \psi_N \bar{\psi}_* (R_*) \exp\left(-\bar{z}^2/2\right)$$

and

$$\rho = \rho_N \bar{\rho}_* (R_*) \exp\left(-\bar{z}^2\right),$$

the equation becomes

$$\Omega_0^2 \left[\frac{\partial \rho}{\partial R} - 6\frac{(R-R_0)}{\Delta_z^2}\rho\right] \simeq \frac{1}{4\pi R_0^2 \Delta_z^2} \times \left\{\psi \frac{\partial}{\partial R}\left[\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{\Delta_z^2}\left(\bar{z}^2 - 1\right)\psi\right] - \left(\frac{\partial \psi}{\partial R}\right)\left[\frac{\partial^2}{\partial R^2} + \frac{1}{\Delta_z^2}\left(\bar{z}^2 - 1\right)\right]\psi\right\}. \quad (41)$$

Master Equation for Local Rigid Rotor Configurations (cont.)

Then we obtain, for $\Delta_R^2/\Delta_z^2 \ll 1$,

$$\frac{d\bar{\rho}_*}{dR_*} \simeq \frac{d}{dR_*}\left[\bar{\psi}_* \frac{d^2\bar{\psi}_*}{dR_*^2} - \left(\frac{d\bar{\psi}_*}{dR_*}\right)^2\right]. \tag{42}$$

Consequently, the relationship between ρ and ψ can be expressed as

$$\bar{\rho}_* = \bar{\rho}_0 - \left(\frac{d\bar{\psi}_*}{dR_*}\right)^2 + \bar{\psi}_* \frac{d^2\bar{\psi}_*}{dR_*^2}. \tag{43}$$

Clearly, in the considered limit $\left(\Delta_R^2 \ll \Delta_z\right)$,

$$\frac{\partial p_M}{\partial \bar{z}^2} \simeq \frac{\psi_N^2}{8\pi R_0^2 \Delta_R^2} \exp\left(-\bar{z}^2\right) \bar{\psi}_*\left(R_*\right) \frac{d^2\bar{\psi}_*}{dR_*^2}. \tag{44}$$

Master Equation for Local Rigid Rotor Configurations (cont.)

Therefore,

$$p_M \simeq -\frac{\psi_N^2}{8\pi R_0^2 \Delta_R^2} \bar{\psi}_* \frac{d^2 \bar{\psi}}{dR_*^2} \exp\left(-\bar{z}^2\right) \tag{45}$$

and if $\bar{\psi}_* d^2\psi_*/dR_*^2 < 0$ we require that

$$p_{G0}\bar{\rho}_* > \frac{\psi_N^2}{8\pi R_0^2 \Delta_R^2} \left(\bar{\psi}_* \frac{d^2\psi_*}{dR_*^2}\right). \tag{46}$$

The relevant toroidal current density is about

$$\begin{aligned} J_\phi &\simeq -\frac{c}{4\pi R_0}\left(\frac{1}{\Delta_R^2}\frac{\partial^2\psi}{\partial R_*^2} + \frac{1}{\Delta_z^2}\frac{\partial^2\psi}{\partial \bar{z}^2}\right) \\ &= -\frac{c\psi_N}{4\pi R_0 \Delta_R^2}\left[\frac{d^2\bar{\psi}_*}{dR_*^2} + \frac{\Delta_R^2}{\Delta_z^2}\left(\bar{z}^2 - 1\right)\bar{\psi}_*\right]\exp\left(-\bar{z}^2/2\right) \end{aligned}$$

Magnetic Field Configurations with even $\bar{\psi}_*(R_*)$

An important set of field and plasma configurations that has not been looked for until now is that for which the magnet surface function ψ is, locally, an even function of R_* and for which the pre-existence of a seed field is not required. For this we refer to rotation frequencies that are independent of ψ ("locally rigid rotors"). Thus the onset of these configurations may be considered as a candidate process for the generation of magnetic fields associated with the combined product of the gravitational force vertical component and of the local plasma density gradient.
In particular, we observe that the function $\sin k_* R_*$ and $\cos k_* R_*$ are solutions of Eq. (44) for $\bar{\rho}_* = 0$ as in this case

$$\frac{d}{dR_*}\left[\bar{\psi}_* \frac{d^2\bar{\psi}_*}{dR_*^2} - \left(\frac{d\bar{\psi}_*}{dR_*}\right)^2\right] = 0. \tag{47}$$

Magnetic Field Configurations with even $\bar{\psi}_* (R_*)$ (cont.)

Therefore, we may consider a periodic solution of the form

$$\bar{\psi}_* \simeq \cos R_* + \frac{\alpha_*}{4} \cos (2R_*) \, ,$$

that, given Eq. (45), can lead to

$$\bar{\rho}_* = \alpha_* \left\{ \frac{5}{4} - \cos R_* \left(\frac{5}{4} - \frac{9}{2} \sin^2 R_* \right) \right\} .$$

Clearly, this is, again, a periodically modulated density profile.
For this class of configurations we may consider also the possibility that Solitary Rings may emerge. As an example of these we take

$$\bar{\psi}_* = \left(1 + R_*^2\right)^{1/2} ,$$

and we note that $d\bar{\psi}_*/dR_* = R_*/\left(1 + R_*^2\right)^{1/2}$. Thus

$$\bar{\rho}_* \simeq \frac{2}{1 + R_*^2} .$$

Locally Differential Rotator Configurations

In this case we assume that a seed magnetic field $B_0\mathbf{e_z}$ is present corresponding to a magnetic surface function $\psi_0 \simeq B_0 R_0 R$, and refer to a surface function $\psi \simeq \psi_0 + \psi_1$ with $|\psi_1| < \psi_0$ but $B_z \sim |\psi_1| / (\Delta_R R_0) > B_0 \sim \psi_0 / R_0^2$.

Then we take

$$V_\phi \simeq \Omega(\psi) R \simeq \Omega_k(R_0) R_0 + \Omega_k(R_0)(R - R_0) + \left(\frac{d\Omega_k}{dR}\right)_{R=R_0} \left[(R - R_0) + \psi_1 / (d\psi_0 / dR)\right] \tag{5}$$

and define

$$\delta\Omega = \left[\frac{d\Omega_K}{dR} \Big/ \frac{d\psi_0}{dR}\right]_{R=R_0} \psi_1 .$$

Locally Differential Rotator Configurations (cont.)

In this case considering the asymptotic limit where $R_0\Delta_R > \Delta_z^2 > \Delta_R^2$ the Master Equation reduces to

$$-\Omega_D^2 R_0 \frac{\partial}{\partial z}\left(\rho\frac{\psi_1}{\psi_0}\right) \simeq \frac{1}{4\pi R_0^2}\left\{\left[\left(\frac{\partial^3}{\partial z^3}\psi_1\right)\frac{\partial\psi_1}{\partial R} - \left(\frac{\partial^3}{\partial R^3}\psi_1\right)\frac{\partial\psi_1}{\partial z}\right] + \left[\left(\frac{\partial}{\partial z}\frac{\partial^2}{\partial R^2}\psi_1\right)\frac{\partial\psi_1}{\partial R} - \left(\frac{\partial}{\partial R}\frac{\partial^2}{\partial z^2}\psi_1\right)\frac{\partial\psi_1}{\partial z}\right]\right\} \quad (6)$$

where

$$\Omega_D^2 = -R_0\frac{d}{dR}\Omega_k^2 = 3\Omega_0^2. \quad (7)$$

Clearly the symmetries of this equation indicate that $\delta\Omega$ and therefore ψ_1 have to be odd functions of R_*. We observe that $\rho\Omega_D^2$ is the driving factor for the field configuration represented by ψ_1. Then we note that the scale distance Δ_z

Locally Differential Rotator Configurations (cont.)

does not affect Eq. (6) in the limit where Δ_R^2/Δ_z^2 can be neglected and in this case, that was analyzed earlier (Coppi and Rousseau, 2006),

$$\frac{\Delta_R}{R_0} \sim \left(\frac{\psi_1 \psi_0}{4\pi\rho R_0^6 \Omega_D^2} \right)^{1/3}. \tag{8}$$

Periodic Ring Structures for Local Differentially Rotating Configurations

These structures can be found in the limit where the ratio Δ_R^2/Δ_z^2 can be considered as negligibly small. Thus they may be viewed as having a "microscopic" radial modulation while the kind of structure identified in Section 7 can have a "macroscopic" radial thickness of the order of Δ_z. In the present case case the Master Equation reduces to that derived already in (Coppi and Rousseau, 2006),

$$\bar{\rho}_* \bar{\psi}_* = \frac{d^3 \bar{\psi}_*}{dR_*^3} \bar{\psi}_* - \frac{d^2 \bar{\psi}_*}{dR_*^2} \frac{d\bar{\psi}_*}{dR_*}, \tag{37}$$

and the considered solution is

$$\bar{\psi}_* = \sin R_* + \frac{\varepsilon_*}{2} \sin 2R_* \,, \tag{38}$$

Periodic Ring Structures for Local Differentially Rotating Configurations (cont.)

leading to the following radial density profile

$$\bar{\rho}_* = \varepsilon_* \frac{\sin^2 R_*}{1 + \varepsilon_* \cos R_*}\,, \tag{39}$$

where the dimensionless parameter $\varepsilon_* \leq 1/4$. We observe that $\bar{\psi}_* = \sin R_*$ is the solution corresponding to $\varepsilon_* = 0$ and to a vanishing modulated density ρ_*. Then the question that remains to be investigated is whether this periodic solution will survive when appreciable values for Δ_R^2/Δ_z^2 are considered and the assumption of separability cannot be maintained. Moreover, we observe that

$$p_M \simeq \frac{1}{8\pi R_0^2}\left\{\frac{1}{{\Delta_R}^2}\psi^2 - I^2\right\} \tag{40}$$

and given the expression (42) for $\bar{\rho}_*$ if we take $I^2 \lesssim (\psi/\Delta_R)^2$ we have no evident problems with the expression for $T_M = m_i p_M/(2\rho)$.

Solitary Ring Solution

A radially localized solution of the Master Equation (6) that involves a Gaussian function is

$$\bar{\psi}_* = R_* \exp\left(-\frac{1}{2}R_*^2\right). \tag{26}$$

The expression for $\bar{\rho}$ that we obtain in this case is

$$\bar{\rho}_* = 2\left[\frac{\Delta_R^2}{\Delta_z^2} + R_*^2\left(1 - \frac{\Delta_R^2}{\Delta_z^2}\right)\right]\exp\left(-\frac{1}{2}R_*^2\right). \tag{27}$$

requiring that $\Delta_R^2/\Delta_z^2 \leq 1$. We observe that the profile (30) corresponds to a single ring when

$$\frac{2}{3} < \frac{\Delta_R^2}{\Delta_z^2} \leq 1. \tag{28}$$

Solitary Ring Solution (cont.)

Clearly, $B_R \sim B_z$ when $\Delta_R \sim \Delta_z$ and in this case $\mathrm{v}_{_A}^2 \sim \mathrm{v}_\phi^0 \mathrm{v}_{th}^0$ where $\mathrm{v}_{_\mathrm{A}}$ is the Alfvén velocity for $\psi_{_N} \sim \psi_0$, $\mathrm{v}_\phi = \Omega_k R_0$ and $\mathrm{v}_{th}^0 \sim (2T_{G0}/m_i)^{1/2}$.
Thus we may argue that the pair of rings collapses into one ring as $(\Delta_R/\Delta_z)^2$ is increased. The poloidal magnetic field components now are

$$B_z \simeq \frac{\psi_{_N}}{R_0 \Delta_R} \left(1 + R_*^2\right) \exp\left[-\frac{1}{2}\left(R_*^2 + \bar{z}^2\right)\right], \tag{29}$$

and

$$B_R \simeq \frac{\psi_{_N}}{R_0 \Delta_R} \bar{z} \bar{R}_* \exp\left[-\frac{1}{2}\left(R_*^2 + \bar{z}^2\right)\right], \tag{30}$$

Thus the relevant magnetic surfaces exhibit two O-points, at $R_* = \pm 1$ and $\bar{z} = 0$. The current density J_ϕ becomes

$$J_\phi \simeq \frac{c\psi_{_N}^0}{4\pi R_0} \exp\left[-\frac{1}{2}\left(R_*^2 + \bar{z}^2\right)\right] R_* \left[\left(1 - \bar{z}^2\right)\frac{1}{\Delta_z^2} + \left(3 - R_*^2\right)\frac{1}{\Delta_R^2}\right] \tag{31}$$

Solitary Ring Solution (cont.)

and of opposite current channels for $R_* > 0$ and $R_* < 0$.

Now if we refer to Eq. (21) where we take $I = 0$ we have

$$p_M = -\frac{R_*^2 \psi_N^2}{8\pi R_0^2} \left[\frac{1}{\Delta^2{}_z} \bar{z}^2 + \frac{1}{\Delta^2{}_R} \left(R_*^2 - 3 \right) \right] \exp\left[- \left(R_*^2 + \bar{z}^2 \right) \right] \tag{32}$$

that is always negative for $R^2{}_* > 3$. On the other hand in this case

$$p_G = p_{NG} \left[\frac{\Delta^2{}_R}{\Delta^2{}_z} + R_*^2 \left(1 - \frac{\Delta^2{}_R}{\Delta^2{}_z} \right) \right] \exp\left[-\frac{1}{2} \left(R_*^2 + \bar{z}^2 \right) \right]. \tag{33}$$

Therefore,

$$\begin{aligned} p \quad = \quad & p_G + p_M = P_{NG} \exp\left[-\frac{1}{2} \left(R_*^2 + \bar{z}^2 \right) \right] \\ & \cdot \left\{ R_*^2 + \frac{\Delta_R^2}{\Delta_z^2} \left(1 - R_*^2 \right) \right. \\ & \left. + \left(\frac{\psi_N^2}{8\pi R_*^2 \Delta_R^2 p_{NG}} \right) \exp\left[-\frac{1}{2} \left(R_*^2 + \bar{z}^2 \right) \right] \left[R_*^2 \left(\rho - R_*^2 \right) - \frac{\Delta_R^2}{\Delta_z^2} \bar{z}^2 \right] \right\}. \end{aligned} \tag{34}$$

Solitary Ring Solution (cont.)

Clearly, for adequate values of p_{NG} the total plasma pressure can remain positive and with a finite total temperature T for all values of $\bar{z}$ and R_*. We observe that the "complex" toroidal current density pattern represented by Eq. (34) is reminiscent of that found for the modes identified in (Coppi, 2006). We note that the complete magnetic surface function ψ is, in this case,

$$\psi = \psi_1 \left(R_*, \bar{z}\right) + R_* \frac{\Delta_R}{R_0} \psi_0 = \psi_N \left[\bar{\psi}_* \left(R_*\right) \exp\left(-\frac{1}{2}\bar{z}^2\right) + R_* \frac{\Delta_R}{R_0} \frac{\psi_0}{\psi_N} \right] \quad (35)$$

and we consider $\psi_N/\Delta_R > \psi_0/R_0$. Therefore, the relevant magnetic surfaces are represented by

$$R_* \exp\left[-\frac{1}{2}\left(R_*^2 + \bar{z}^2\right)\right] + \varepsilon_z R_* = \text{constant.} \quad (36)$$

for $\varepsilon_z < 1$, referring to Eq. (19). A graphical representation of these surfaces is given in the following figure.

Solitary Ring Solution (cont.)

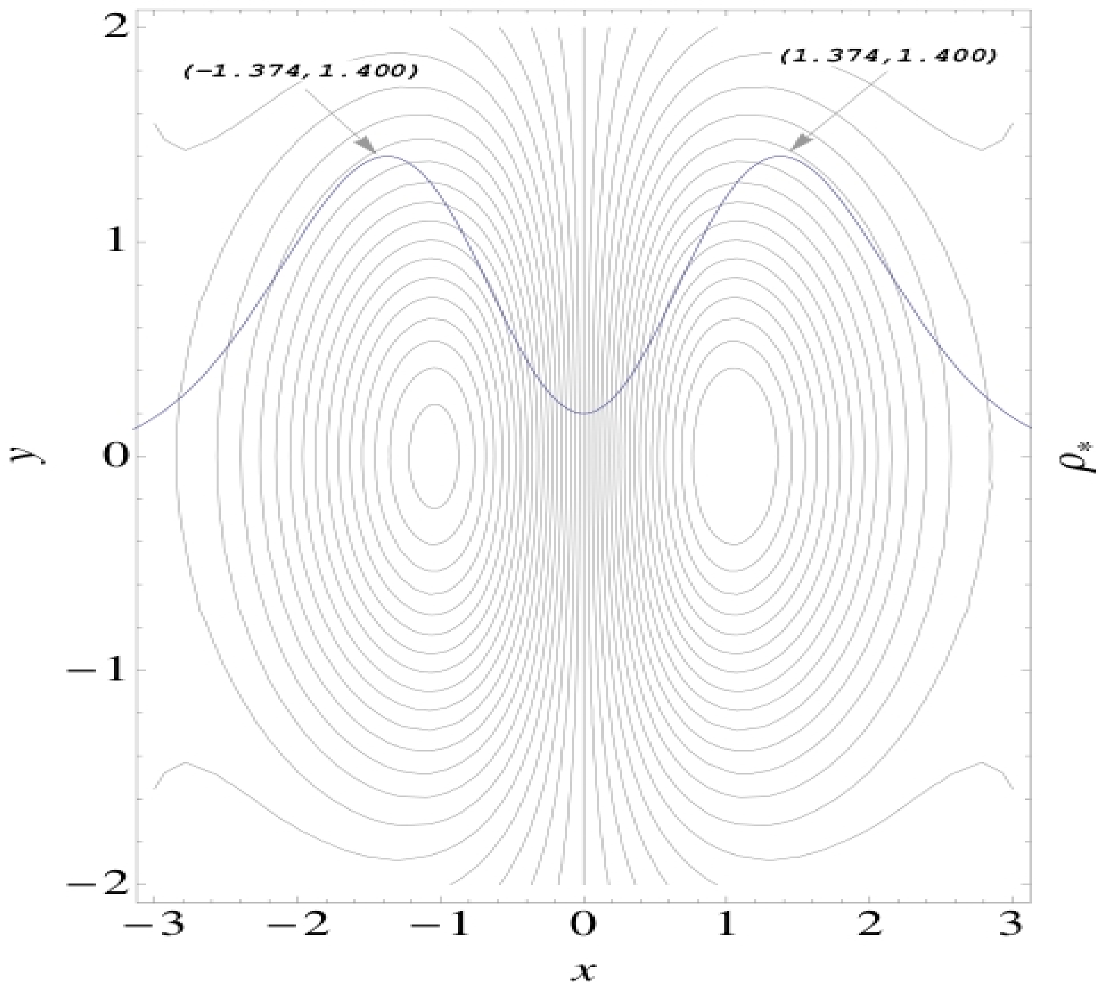


Figure: Graphical representation of the magnetic surfaces for the configuration corresponding to Eq. (38). The curve with dotted heavy lines indicates the single ring density profiles represented by Eq. (39) for $\Delta_R^2/\Delta_z^2 = 1/10$.

Vertical Momentum Density Conservation

The vertical equilibrium equation connects the total plasma pressure to the particle density and the magnetic field configuration. In particular this equation can be written as

$$0 \simeq -\Omega_k^2 \rho z - \frac{\partial}{\partial z}\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\left(\mathbf{B} \cdot \nabla \mathbf{B}\right)_z \quad (9)$$

and we find it convenient to separate p into $p_{_G} + p_{_M}$ where

$$2\frac{\partial}{\partial z^2} p_{_G} = -\Omega_k^2 \rho. \quad (10)$$

Thus we may define the temperature

$$T_G = \frac{m_i}{2}\frac{p_{_G}}{\rho} \quad (11)$$

Vertical Momentum Density Conservation (cont.)

where m_i is the mass of the nuclei of which the plasma is composed and consider sufficiently high values of T_G that $p = p_G + p_M$ is always positive. Clearly

$$\frac{\partial p_M}{\partial z} = -\frac{1}{8\pi}\frac{\partial}{\partial z}\left(B_R^2 + B_\phi^2\right) - \frac{1}{4\pi}\left(B_R\frac{\partial}{\partial R}B_z\right). \tag{12}$$

If we consider axisymmetric configurations for which $I = I(\psi)$ this reduces to

$$\frac{\partial p_M}{\partial z} = -\frac{1}{4\pi R^2}\frac{\partial \psi}{\partial z}\left(\Delta_* \psi + I\frac{dI}{d\psi}\right) \tag{13}$$

leading to the so called "G-S equation" for magnetically confined plasmas where $\nabla p = (dp/d\psi)\nabla\psi$ and gravity and rotation are not included. The so called "pulsar equation" describing the magnetic configuration of an axisymmetric plasma surrounding a rotating neutron star was derived first and

Vertical Momentum Density Conservation (cont.)

solved along similar lines by Cohen, Coppi and Treves, 1973. In the present case we do not consider $p_M = p_M(\psi)$, and have, for configurations localized around $R = R_0$,

$$\frac{\partial}{\partial z}\left\{p_M + \frac{1}{8\pi R_0^2}\left[I^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]\right\} \simeq -\frac{1}{4\pi R_0^2}\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial R^2}. \tag{14}$$

In particular, if introduce the dimensionless variables R_* and $\bar{z}$, Eq. (17) becomes

$$\frac{\partial}{\partial\bar{z}}\left\{p_M + \frac{1}{8\pi R_0^2}\left[I^2 + \frac{1}{\Delta_z^2}\left(\frac{\partial\psi}{\partial\bar{z}}\right)^2\right]\right\} \simeq -\frac{1}{4\pi R_0^2\Delta_R^2}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial^2\psi}{\partial R_*^2}. \tag{15}$$

Factorized Solutions

Now we limit our analysis to functions $\bar{\psi}$, that can be factorized as follows

$$\bar{\psi} = \bar{\psi}_* (R_*) \exp\left(-\frac{\bar{z}^2}{2}\right) \tag{16}$$

and Eq. (18) becomes

$$\frac{\partial}{\partial \bar{z}} \left\{ p_M + \frac{I^2}{8\pi R_0^2} \right\} \simeq \frac{1}{8\pi R_0^2} \left\{ \frac{1}{\Delta_z^2} \bar{\psi}_*^2 + \frac{1}{\Delta_R^2} \bar{\psi}_* \frac{d^2 \bar{\psi}_*}{\Delta \bar{R}_*^2} \right\} \exp\left(-\bar{z}^2\right). \tag{17}$$

Therefore

$$p_M + \frac{I^2}{8\pi R_0^2} \simeq -\frac{1}{16\pi R_0^2} \left\{ \frac{1}{\Delta_z^2} \bar{\psi}_*^2 + \frac{1}{\Delta_R^2} \bar{\psi}_* \frac{d^2 \bar{\psi}_*}{d R_*^2} \right\} \exp\left(-\bar{z}^2\right). \tag{18}$$

Then we refer to Eq. (13), assume for simplicity that $T_G = T_{G0}$ is constant, write

$$\frac{4}{m_i} \frac{T_{G0}}{\Omega_k^2} \frac{\partial \rho}{\partial z^2} = -\rho \tag{19}$$

Factorized Solutions (cont.)

and define

$$\Delta_G^2 \equiv \frac{4T_G}{m_i\Omega_*^2}\,. \tag{20}$$

Consequently,

$$\rho = \rho_N \bar{\rho}_* \left(R_*\right) \exp\left(\frac{-z^2}{\Delta_G^2}\right) \tag{21}$$

The relationship between Δ_G^2 and Δ_z^2 will depend on the classes of solutions of the Master Equation that we shall consider. In particular, for the Local Differentially Rotator configurations we shall find

$$\Delta_G^2 = 2\Delta_z^2 \tag{22}$$

and for the Locally Rigid Rotor configurations

$$\Delta_G^2 = \Delta_z^2. \tag{23}$$

Factorized Solutions (cont.)

We observe also that

$$\frac{\partial}{\partial z} I^2 = 2I \frac{\partial \psi}{\partial z} \frac{dI}{d\psi} \tag{24}$$

and in the case where $\psi \simeq \psi_0 + \psi_1$ we have

$$\frac{\partial}{\partial z} I^2 \simeq - \left[\frac{d}{d\psi_0} I^2 \left(\psi_0 \right) \right] \psi_{_N} \bar{\psi}_* \left(R_* \right) \frac{1}{\Delta_z} \exp \left(- \frac{\bar{z}^2}{2} \right) . \tag{25}$$

Therefore in the present case this component of $p_{_M}$ has the same z-profile as $p_{_G}$.