A Triplectic Bi-Darboux Theorem and Para-Hypercomplex Geometry

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Work inspired by: Grigoriev & Semikhatov 1997 & 1998

A Triplectic Bi-Darboux Theorem and Para-Hypercomplex Geometry

Poincaré Lemma

Coordinates

$$x = (x^1, \ldots, x^n) \qquad c = (c^1, \ldots, c^n)$$

x's and c's have opposite Grassmann parity

$$\varepsilon(c^i) = \varepsilon(x^i) + 1$$

Forms

$$\omega = \omega(x, c)$$

A form ω can be viewed as a superfunction of x's and c's

Exterior derivative

$$d = c^i \frac{\partial}{\partial x^i}$$

Poincaré Lemma

Closed

$$d\omega \;=\; 0$$

NB! 0-forms are non-trivial cohomology. No 0-forms allowed.

$$\deg(\omega) \;\geq\; 1 \qquad \omega \;=\; \underbrace{\omega^{(0)}}_{=0} \oplus \omega^{(1)} \oplus \omega^{(2)} \oplus \ldots$$

$$\Downarrow$$

Exact

$$\exists \text{ locally } (r-1)\text{-form } \; \eta = \eta(x,c): \;\; \omega \;=\; d\eta$$

Fine print

- Our proof technique works in the category of (real) analytic superfcts rather than thecategory of smooth C^{∞} superfcts.
- Considers an arbitrary fixed point $x_{(0)}$.
- Restricts to a sufficiently small neighborhood around $x_{(0)}$ if necessarily.
- Assume by change of coordinates that the fixed point $x_{(0)} = 0$ is zero.

Exterior Derivative

Exterior Derivative

$$d \;=\; c^i \frac{\partial}{\partial x^i}$$

Fermionic

$$\varepsilon(d) \;=\; 1$$

1st order

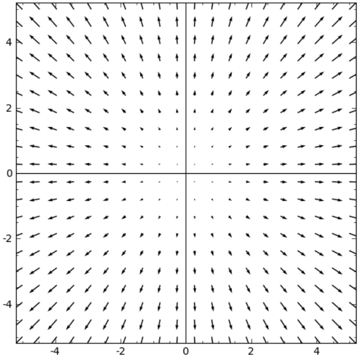
$$\mathrm{order}(d) \;=\; 1$$

Nilpotent

$$2d^2 \;=\; [d,d] \;=\; 0$$

$[A,B] \;=\; AB-(-1)^{\varepsilon_A\varepsilon_B}BA$ denotes the supercommutator.

Euler Vector Field



Euler vector field

$$X = X^i \frac{\partial}{\partial x^i} \qquad X^i = x^i$$

Contraction

Contraction

$$X \lrcorner \;=\; i_X \;=\; i \;=\; x^i \frac{\partial}{\partial c^i}$$

Fermionic

$$\varepsilon(i) \;=\; 1$$

1st order

$$\mathrm{order}(i) \;=\; 1$$

Nilpotent

$$2i^2 \;=\; [i, i] \;=\; 0$$

Fermionic Duality

$$i \;=\; x^i \frac{\partial}{\partial c^i} \qquad \text{is dual to} \qquad d \;=\; c^i \frac{\partial}{\partial x^i}$$

Lie Derivative

Lie derivative

$$\mathcal{L}_X = \mathcal{L} = [d,i] = x^i \frac{\partial}{\partial x^i} + c^i \frac{\partial}{\partial c^i} = N_x + N_c$$

Lie derivative

$$\begin{aligned}
\mathcal{L} &= [d,i] = [d, x^i \frac{\partial}{\partial c^i}] \\
&= [d,x^i]\frac{\partial}{\partial c^i} + x^i [d, \frac{\partial}{\partial c^i}] \\
&= [d,x^i]\frac{\partial}{\partial c^i} + x^i [\frac{\partial}{\partial c^i}, d] \\
&= c^i \frac{\partial}{\partial c^i} + x^i \frac{\partial}{\partial x^i} \\
&= N_c + N_x
\end{aligned}$$

$$d = c^i \frac{\partial}{\partial x^i}$$

Super Euler vector field

Bosonic $\varepsilon(\mathcal{L}) = 0$ $\mathrm{order}(\mathcal{L}) = 1$

Lie Derivative

Lie Derivative as Super Euler vector field

$$\mathcal{L}\omega(x,c) \;=\; (N_x + N_c)\omega(x,c)$$

Contraction/Homotopy Op

$$\mathcal{L}^{-1}\omega(x,c) \;=\; \frac{1}{N_x + N_c}\omega(x,c) \;=\; \int_0^1 \frac{dt}{t}\omega(tx,tc)$$

$$\int_0^1 dt\; t^n \;=\; \frac{1}{n+1} \qquad\qquad \int_0^1 \frac{dt}{t}\; t^n \;=\; \frac{1}{n}$$

Poincaré Lemma

Closed

$$\omega = \omega(x, c) \qquad d\omega = 0$$

No 0-form allowed

$$\deg(\omega) \geq 1 \qquad \omega = \underbrace{\omega^{(0)}}_{=0} \oplus \omega^{(1)} \oplus \omega^{(2)} \oplus \dots$$

$$\text{def} \qquad \eta = i\mathcal{L}^{-1}\omega = \mathcal{L}^{-1}i\omega$$

Proof

$$d\eta = d\mathcal{L}^{-1}i\omega = \mathcal{L}^{-1}di\omega = \mathcal{L}^{-1}[d, i]\omega = \mathcal{L}^{-1}\mathcal{L}\omega = \omega \text{ exact}$$

A Triplectic Bi-Darboux Theorem and Para-Hypercomplex Geometry

Coordinates

Triple

$$x \;=\; (x^1,\ldots,x^n) \qquad y \;=\; (y^1,\ldots,y^n) \qquad c \;=\; (c^1,\ldots,c^n)$$

c's have opposite Grassmann parity of the x's and y's

$$\varepsilon(x^i) \;=\; \varepsilon(y^i) \;=\; \varepsilon(c^i)+1$$

To not clog slides with Grassmann sign factors, let us simplify:

Bosonic	Bosonic	Fermionic
$\varepsilon(x^i) \;=\; 0$	$\varepsilon(y^i) \;=\; 0$	$\varepsilon(c^i) \;=\; 1$

The theory works more generally in a superized formalism.

Two Exterior Derivatices

Exterior Derivatives

$$d^1 = c^i \frac{\partial}{\partial x^i} \qquad d^2 = c^i \frac{\partial}{\partial y^i} \qquad d = d^1 d^2 \qquad \text{2nd order}$$

$$\text{Fermionic} \qquad \varepsilon(d^1) = 1 = \varepsilon(d^2) \qquad \varepsilon(d) = 0 \qquad \text{Bosonic}$$

Supercommute

$$(d^1)^2 = 0 \qquad (d^2)^2 = 0 \qquad d^1 d^2 + d^2 d^1 = 0$$

$$[d^a, d^b] = 0 \qquad a, b \in \{1, 2\}$$

Closedness Relations

$$f = \frac{1}{2} f_{ij}(x,y) c^i c^j \qquad \text{2-form} \qquad f_{ji} = -f_{ij}$$

closed $\quad d^1 f = 0 \quad \Leftrightarrow \quad \sum_{\text{cycl. } i,j,k} \frac{\partial f_{jk}(x,y)}{\partial x^i} = 0$

closed $\quad d^2 f = 0 \quad \Leftrightarrow \quad \sum_{\text{cycl. } i,j,k} \frac{\partial f_{jk}(x,y)}{\partial y^i} = 0$

What is the most general solution to f locally?

$$\exists \text{ loc. 0-form } g = g(x,y) : \; f = dg \text{ exact } \; \Leftrightarrow \; f_{ij} = \frac{\partial^2 g(x,y)}{\partial x^i \partial y^j} - (i \leftrightarrow j)$$

Bi-Poincaré Lemma

$$\omega = \omega(x, y, c) \qquad\qquad d = d^1 d^2$$

Closed

$$d^1\omega = 0 = d^2\omega$$

Exact

$$\exists \text{ locally form } \eta = \eta(x, y, c) : \quad \omega = d\eta \qquad \text{exact}$$

NB! 0- and 1-forms are non-trivial cohomology. No 0- and 1-forms allowed.

$$\deg(\omega) \geq 2 \qquad \omega = \underbrace{\omega^{(0)}}_{=0} \oplus \underbrace{\omega^{(1)}}_{=0} \oplus \omega^{(2)} \oplus \dots$$

Two Contractions

Contractions

$$i_1 = x^i \frac{\partial}{\partial c^i} \qquad i_2 = y^i \frac{\partial}{\partial c^i} \qquad i = i_2 i_1 \qquad \text{2nd order}$$

Fermionic $\quad \varepsilon(i_1) = 1 = \varepsilon(i_2) \qquad \varepsilon(i) = 0 \quad$ Bosonic

Supercommute

$$(i_1)^2 = 0 \qquad (i_1)^2 = 0 \qquad i_1 i_2 + i_2 i_1 = 0$$

$$[i_a, i_b] = 0 \qquad a, b \in \{1, 2\}$$

Four Lie Derivatives

Lie derivatives

$$\mathcal{L}^a_b \;=\; [d^a, i_b] \qquad a, b \in \{1, 2\}$$

Bosonic $\qquad \varepsilon(\mathcal{L}^a_b) \;=\; 0$

$$\mathcal{L}^1_1 \;=\; N_x + N_c \qquad \mathcal{L}^2_2 \;=\; N_y + N_c \qquad \leftarrow \text{Diagonal}$$

$$N_x \;=\; x^i \frac{\partial}{\partial x^i} \qquad N_y \;=\; y^i \frac{\partial}{\partial y^i} \qquad N_c \;=\; c^i \frac{\partial}{\partial c^i}$$

$$\mathcal{L}^2_1 \;=\; x^i \frac{\partial}{\partial y^i} \;=\; J_+ \qquad \mathcal{L}^1_2 \;=\; y^i \frac{\partial}{\partial x^i} \;=\; J_- \qquad \leftarrow \text{Not diagonal}$$

QM paradigm: Look for max. com. set of observables!

Lie Algebras

$gl(2,\mathbb{C})$ Lie alg

$$[\mathcal{L}^a_b, \mathcal{L}^c_d] = \delta^a_d \mathcal{L}^c_b - \delta^c_b \mathcal{L}^a_d$$

$$\underbrace{gl(2,\mathbb{C})}_{\mathcal{L}^a_b} = \underbrace{sl(2,\mathbb{C})}_{J_\alpha} \oplus \underbrace{\mathbb{C}}_{\mathcal{L}}$$

$$J_1 = \frac{\mathcal{L}^2_1 + \mathcal{L}^1_2}{2} \qquad J_2 = \frac{\mathcal{L}^2_1 - \mathcal{L}^1_2}{2\mathrm{i}} \qquad J_3 = \frac{\mathcal{L}^1_1 - \mathcal{L}^2_2}{2} = \frac{N_x - N_y}{2}$$

$sl(2,\mathbb{C})$ Lie alg

$$[J_\alpha, J_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma} J_\gamma \qquad \alpha,\beta,\gamma \in \{1,2,3\} \qquad \varepsilon_{123} = 1$$

$$\mathcal{L} = \mathcal{L}^a_a = N_x + N_y + 2N_c$$

$\mathcal{L}$ Casimir

$$[\mathcal{L}, \mathcal{L}^a_b] = 0$$

Bi-Poincaré Lemma Strategy

$$d = d^1 d^2 \qquad i = i_2 i_1 \qquad \text{2nd order}$$

Def

$$\text{3rd ord. } L = [d, i] = \ldots = \Lambda + (\ldots)_b d^b$$

$$\text{2nd ord. } \Lambda = \frac{\mathcal{L}}{2}\left(\frac{\mathcal{L}}{2}+1\right) - \underbrace{J^2}_{J_1^2+J_2^2+J_3^2}$$

$$[L, \mathcal{L}^a_b] = 0 \qquad \text{Casimir}$$

$$[\Lambda, \mathcal{L}^a_b] = 0 \qquad \text{Casimir}$$

Assumption

Assume Λ^{-1} exists

$$\text{closed} \quad d^a\omega = 0 \quad a \in \{1,2\} \qquad \text{def} \quad \eta = i\Lambda^{-1}\omega$$

Proof

$$\begin{aligned} d\eta &= di\Lambda^{-1}\omega = (L + id)\Lambda^{-1}\omega = \Lambda^{-1}L + id\Lambda^{-1}\omega \\ &= \Lambda^{-1}(\Lambda + (\ldots)_b d^b) + i\Lambda'^{-1}d\omega = \omega \qquad \text{exact} \end{aligned}$$

Algebra of Forms

$$\begin{aligned}\mathcal{A} &= \mathcal{A}[[x,y,z]]] = \{\omega = \omega(x,y,c)\} \\ &= \bigoplus_{n_x,n_y,n_c=0}^{\infty} \mathcal{A}_{n_x,n_y,n_c} \qquad \infty \text{ dim vector space}\end{aligned}$$

Form

$$\omega = \bigoplus_{n_x,n_y,n_c=0}^{\infty} \omega^{(n_x,n_y,n_c)}$$

small letter=eigenvalues

$$\begin{aligned}n_x &= \text{eigenvalue of } N_x = x^i \frac{\partial}{\partial x^i} \\ n_y &= \text{eigenvalue of } N_y = y^i \frac{\partial}{\partial y^i} \\ n_c &= \text{eigenvalue of } N_c = c^i \frac{\partial}{\partial c^i}\end{aligned}$$

Capital Letter=Operator

Algebra of Forms as $gl(2,\mathbb{C})$ Rep

- Alg. of forms $\quad\leftrightarrow\quad$ Hilbert space of states

$$\mathcal{A} \;=\; \mathcal{A}[[x,y,c]]$$

- Constant zero-form $\quad\leftrightarrow\quad$ vacuum

$$1 \;=\; |0\rangle \;=\; \Omega$$

- Creation op

$$x^i \qquad y^j \qquad c^k$$

- Annihilation op

$$\frac{\partial}{\partial x^i} \qquad \frac{\partial}{\partial y^j} \qquad \frac{\partial}{\partial c^k}$$

- Generators $\mathcal{L}^a_b$ act on $\mathcal{A}$ $\qquad$ $\mathcal{A}$ is ∞-dim rep

Good Quantum Numbers n_{xy} and n_c

$$\mathcal{A} = \bigoplus_{n_{xy},n_c=0}^{\infty} \mathcal{A}_{n_{xy},n_c} \qquad \infty \text{ dim vector space}$$

Form

$$\omega = \bigoplus_{n_{xy},n_c=0}^{\infty} \omega^{(n_{xy},n_c)}$$

$$n_{xy} = \text{eigenvalue of } N_x + N_y$$
$$n_c = \text{eigenvalue of } N_c$$

$$[N_x + N_y, \mathcal{L}^a_b] = 0 \qquad [N_c, \mathcal{L}^a_b] = 0 \qquad \text{Casimirs}$$

Let n_{xy} and n_c be fixed numbers.

Generators $\mathcal{L}^a_b$ act on $\mathcal{A}_{n_{xy},n_c}$ $\qquad$ $\mathcal{A}_{n_{xy},n_c}$ is finite-dim rep

Fixed $\mathcal{A}_{n_{xy},n_c}$

$$n_{xy} = \text{eigenvalue of } N_x + N_y$$
$$n_c = \text{eigenvalue of } N_c$$
$$n_{xy} + 2n_c = \ell = \text{eigenvalue of } \mathcal{L} \qquad \mathcal{L} = N_x + N_y + 2N_c$$

$sl(2,\mathbb{C})$ Representation Theory

Finite dim $\Rightarrow$ completely reducible

$$\mathcal{A}_{n_{xy},n_c} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}_0} \mu_j V_j$$

$V_j = sl(2,\mathbb{C})$ irrep $\qquad \mu_j \in \mathbb{N}_0$ multiplicity

Strategy: Enough to study:

Alg. of forms

$$\mathcal{A} = \bigoplus_{n_{xy}, n_c = 0}^{\infty} \mathcal{A}_{n_{xy}, n_c} \qquad \infty \text{ dim rep}$$

$$\cup$$

Fixed good quantum numbers n_{xy}, n_c, ℓ

$$\mathcal{A}_{n_{xy}, n_c} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}_0} \mu_j V_j \qquad \text{finite-dim rep}$$

$$\cup$$

Fixed more good quantum numbers j, λ

Fixed irrep V_j finite-dim irrep

Fixed irrep V_j

$$m = \text{eigenvalue of } J_3 \qquad J_3 = \frac{N_x - N_y}{2} \qquad |m| \leq \frac{n_{xy}}{2}$$

$$j(j+1) = \text{eigenvalue of } J^2 \qquad m \in \{-j, 1-j, \ldots, j-1, j\} \qquad j \leq \frac{n_{xy}}{2}$$

$$\lambda = \text{eigenvalue of } \Lambda \qquad \Lambda = \frac{\mathcal{L}}{2}\left(\frac{\mathcal{L}}{2}+1\right) - J^2$$

Proof

$$\begin{aligned} \lambda &= \frac{\ell}{2}\left(\frac{\ell}{2}+1\right) - j(j+1) \\ &\geq \left(\frac{n_{xy}}{2}+n_c\right)\left(\frac{n_{xy}}{2}+n_c+1\right) - \frac{n_{xy}}{2}\left(\frac{n_{xy}}{2}+1\right) \\ &= (n_{xy}+n_c)\underbrace{(n_c-1)}_{>0} > 0 \qquad \text{because } n_c = \deg(\omega) \geq 2 \end{aligned}$$

Bi-Poincaré Lemma

Lemma

Λ is diagonalizable with $\mathrm{Spec}(\Lambda) > 0$ on forms ω with $\deg(\omega) \geq 2$.

Bi-Poincaré Lemma

$$\left.\begin{array}{rcl} d^1\omega & = & 0 \\ d^2\omega & = & 0 \\ \deg(\omega) & \geq & 2 \end{array}\right\} \quad \Rightarrow \quad \text{locally } \omega = d\eta \text{ exact}$$

$$d = d^1 d^2$$

A Triplectic Bi-Darboux Theorem and Para-Hypercomplex Geometry

Complex Hodge Theory: Dolbeault Op

form $\omega = \omega(z, \bar{z}, dz, d\bar{z})$ Dolbeault op $[\partial, \bar{\partial}] = 0,\ \partial^2 = 0, \bar{\partial}^2 = 0.$

Poincaré Lemma

$$\bar{\partial}\omega = 0 \quad \Rightarrow \quad \text{locally } \omega = \underbrace{\bar{\partial}\eta}_{\text{exact}} + \underbrace{f(z, dz)}_{\text{hol. form}}$$

$$\eta = \bar{i}\bar{\mathcal{L}}^{-1}\omega$$

Bi-Poincaré Lemma

$$\left.\begin{aligned} \partial\omega &= 0 \\ \bar{\partial}\omega &= 0 \end{aligned}\right\} \Rightarrow \text{loc. } \omega = \underbrace{\partial\bar{\partial}\eta}_{\text{exact}} + \underbrace{f(z, dz)}_{\text{hol. form}} + \underbrace{g(\bar{z}, d\bar{z})}_{\text{antihol. form}}$$

$$\eta = \bar{i}i\bar{\mathcal{L}}^{-1}\mathcal{L}^{-1}\omega$$

Real Hodge Theory

$$\text{form} \qquad \omega = \omega(x,c) \qquad\qquad c^i = dx^i$$

$$\text{ext. deriv. } d = c^i \frac{\partial}{\partial x^i} \qquad\qquad \text{1st order}$$

$$\text{adj. } \star d \star \sim d^\dagger = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial c^j} \qquad \text{2nd order BV odd Lapl.}$$

$$d^2 = 0 \qquad (d^\dagger)^2 = 0 \qquad [d^\dagger, d] = \Delta \text{ Beltrami Lapl.}$$

Bi-Poincaré Lemma

$$\left.\begin{array}{rcl} d\omega & = & 0 \\ d^\dagger\omega & = & 0 \end{array}\right\} \quad \Rightarrow \quad \text{locally } \omega = \Delta\eta \text{ exact}$$

A Triplectic Bi-Darboux Theorem and Para-Hypercomplex Geometry

Poisson Manifold with Local Coordinates

- Manifold $\mathcal{M}$.
- Poisson bracket $\{\cdot,\cdot\}$.
- PB has intrisic Grassmann parity $\varepsilon \;=\; \begin{cases} 0 & \text{even PB} \\ 1 & \text{odd PB} \end{cases}$
- Locally there exist coordinates z^I of Grassmann parity ε_I.
- Poisson bivector $\pi^{IJ} \;=\; \{z^I, z^J\}$ may depend on z^K.

To not clog slides with Grassmann sign factors, let us simplify:

Bosonic Coordinates

$$\varepsilon(Z^I) \;=\; 0$$

Bosonic PB

$$\varepsilon \;=\; 0$$

The theory works more generally in a superized formalism.

Darboux Theorem

Regular Poisson bivector π^{IJ}. Assume $\mathrm{rank}(\pi^{IJ}) \;=\;$ constant.

Darboux theorem

Locally there exist Bosonic Darboux coordinates:

positions q^i momenta p_j Casimirs c_α

$$\{q^i, p_j\} \;=\; \delta^i_j \;=\; -\{p_j, q^i\}$$

All other fund. PB $= 0$, *i.e.*,

$$\{q^i, q^j\} \;=\; 0 \qquad \{p_i, p_j\} \;=\; 0 \qquad \{c_\alpha, \cdot\} \;=\; 0$$

Morale: Jac. id. are the integrability cond. for $\exists$ Darboux coord.

Two Poisson Brackets

$$\{\cdot,\cdot\}^1 \qquad \{\cdot,\cdot\}^2$$

Compatibility cond = 6-term Mixed Jac Id

$$\sum_{\text{cycl. } f,g,h} \{\{f,g\}^1,h\}^2 = -(1 \leftrightarrow 2)$$

Sym. Jac. id. are the main ammunition for what to follow.

- Used in integrable systems to recursively generate infinitely many conserved charges (Magri's method 1978).
- Used in BRST/anti-BRST triplectic quantization (1995).
- Questions: Does there exists common Darboux coordinates?
- Gelfand and Zakharevich (2000) investigate case with at least one non-deg. bracket.

Triplectic manifold

Def. triplectic manifold $(\mathcal{M}; \{\cdot,\cdot\}^a)$

- $3n$-dimensional manifold $\mathcal{M}$
- equipped with two Poisson brackets $\{\cdot,\cdot\}^1$ and $\{\cdot,\cdot\}^2$
- that both have rank $2n$ out of $3n$ possible,
- that are compatible, *i.e.*, the mixed Jac. id.
- that are jointly non-degenerate, which means that there are no common Casimirs.
- and that have mutually involutive Casimirs, which means that the Casimirs with respect to one bracket are in involution with respect to the other bracket, and vice-versa.

Base manifold $\mathcal{N}$

- Define notation: $c_k \;=\;$ Casimirs for 1st PB.
- Define notation: $p_j \;=\;$ Casimirs for 2nd PB.

Base manifold $\mathcal{N}$

$\mathcal{N} = 2n$ dim manifold of Casimir variables p_j and c_k.

Fiber bundle

$\mathcal{M} \;\rightarrow\; \mathcal{N}$

Two Paracomplex Structures Σ and P

- A complex structure $J: T\mathcal{N} \to T\mathcal{N}$ $\qquad J^2 = -1$
- A paracomplex structure $P: T\mathcal{N} \to T\mathcal{N}$ $\qquad P^2 = 1$
  $=$ local product structure

1st Paracomplex str.

Σ	p_j	c_j
p_i	δ_i^j	0
c_i	0	$-\delta_i^j$

2nd Paracomplex str.

P	p_j	c_j
p_i	0	$(E^{-1})^j{}_i$
c_i	$E^j{}_i$	0

- Sym. Jac. Id. $\Rightarrow$ P integrable
- $\{\Sigma, P\}_+ = 0$ anticommute
- $J := P\Sigma$ complex structure
- Triple (Σ, P, J) para-hypercomplex structure

Para-Hypercomplex Structure

Thm

There is a one-to-one correspondence between triplectic manifolds and twisted para-hypercomplex manifolds

- A para-Hypercomplex manifold is endowed with an Obata connection ∇, *i.e.,* unique torsionfree connection compatible with the para-hypercomplex structure.
- Twisting refers a two-form field F^{ij}.

Hyper-paracomplex

$SO^{+}(2,1;\mathbb{R})$ sym

Bi-Poisson

$SL(2,\mathbb{R})$ sym

Caratheodory-Jacobi-Lie (CJL) Theorem

- Define notation: $c_k \;=\;$ Casimirs for 1st PB.
- Define notation: $p_j \;=\;$ Casimirs for 2nd PB.
- CJL Theorem implies $\exists q^i$ so 1st PB on Darboux form.
- CJL does this **without** changing the c_i's and p_j's.

1 PB $\{\cdot,\cdot\}^1$

	q^j	p_j	c_j
q^i	0	δ^i_j	0
p_i	-δ^j_i	0	0
c_i	0	0	0

2 PB $\{\cdot,\cdot\}^2$

	q^j	p_j	c_j
q^i	F^{ij}	0	$E^i{}_j$
p_i	0	0	0
c_i	-$E^j{}_i$	0	0

Canonical Transformations for 1st PB

- Only two remaining non-trivial matrix structures

$$E^{i}{}_{j} \;=\; \{q^{i}, c_{j}\}^{2} \;=\; E^{i}{}_{j}(p,c) \qquad F^{ij} \;=\; \{q^{i}, q^{j}\}^{2} \;=\; F^{ij}(p,c)$$

- Can we also get 2nd PB on Darboux form **without** spoiling Darboux form for 1st PB?
- Only CT for 1st PB allowed
- c_i are passive spectators
- locally $F_3 = F_3(q', p)$ type CT
- Generator F_3 must be linear in q'

$$-F_3 \;=\; A_j(p)q'^{j} + B(p)$$

Bi-Darboux Theorem

Bi-Darboux Theorem

Necessary and Sufficient condition for Bi-Darboux coordinates on triplectic manifold is that

- (in triplectic language) The ${E^i}_k$ matrix factorizes

$$ {E^i}_k(p,c) \;=\; {P^i}_j(p){C^j}_k(c) $$

- (in para-hypercomplex language) The Obata connection ∇ is flat.

F^{ij} Matrix?

Closed

$F^{ij} = \{q^i, q^j\}^2$ closed because of mixed Jac. id.

$$\sum_{\text{cycl. } i,j,k} \frac{\partial F^{jk}(p,c)}{\partial p_i} = 0$$

$$\sum_{\text{cycl. } i,j,k} \frac{\partial F^{jk}(p,c)}{\partial c_i} = 0$$

- Is it possible to make F^{ij} matrix vanish by CT?

$$F^{ij}(p,c) = \frac{\partial^2 B(p,c)}{\partial c_i \partial p_j} - (i \leftrightarrow j) \qquad \text{exact}$$

- Yes, because of Bi-Poincaré Lemma.

Conclusions

- We have proved a Bi-Poincaré Lemma for triples of variables.
- Rather than the standard method of using Fermionic duality, the new proof relies heavily on $sl(2,\mathbb{C})$ rep. theory; morally a kind of triality.
- We have proved a Bi-Darboux Theorem for triplectic manifolds.
- This strengthen the geometric foundation of triplectic quantization.
- This may infuse renewed interests and developments in triplectic quantization.
- We have proved a one-to-one correspondence between triplectic manifolds and twisted para-hypercomplex manifolds.

References

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